

Determinacy and Analytical Tail Solutions in the Sequence Space

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Abstract

We develop a unified method for assessing local equilibrium determinacy and computing impulse responses in continuous-time heterogeneous-agent models. A forward–backward decomposition of linearized equilibrium dynamics separates stable adjustments propagated forward from components determined by future shocks and infinite-horizon admissibility. Matching these components to inherited states characterizes existence and uniqueness and determines the response when it is unique. Our central constructive result obtains this decomposition from household response operators and market-clearing price feedback. Starting from a known fixed-price household split, we use the response factors underlying fake news kernel and small price-consistency solves to construct the equilibrium stable and unstable projections. This construction makes the common determinacy and solution mechanism operational without requiring a full equilibrium eigendecomposition. The same projections supply an exact continuation condition for a stationary infinite tail, removing time-horizon truncation error.

JEL codes: C61, C63, E21

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1 Introduction

Solving a heterogeneous-agent model after a shock requires paths of household choices, distributions, and prices that clear markets and respect inherited states. The same economic restrictions determine whether a response exists, whether it is unique, and how expectations about the distant future affect current decisions. We develop a continuous-time sequence-space method that treats determinacy and impulse-response computation as parts of one equilibrium solution. Its organizing device is a forward–backward decomposition of the linearized dynamics, constructed from household response operators and market-clearing price feedback.

Sequence-space methods are especially useful when individual heterogeneity creates a large distributional state but equilibrium depends on comparatively few prices and aggregates (Auclert et al., 2021). The sequence-space Jacobian (SSJ) describes the response of equilibrium residuals to entire paths of aggregate variables. Those paths extend over an unbounded time interval, even when the number of aggregate variables is small. A finite-horizon calculation must therefore account for the continuation after its terminal date. An invertible truncated Jacobian does not by itself establish existence and uniqueness on the infinite horizon, and a small residual within the computed interval does not ensure that the implied continuation is admissible.

Our solution mechanism starts from the coupled Hamilton–Jacobi–Bellman (HJB), Kolmogorov forward (KFE), and market-clearing equations. After regular static constraints are eliminated, the linearized equilibrium dynamics can, under the stated separation conditions, be decomposed into stable and unstable components. The stable component propagates an initial adjustment forward and accumulates subsequent shocks. Infinite-horizon admissibility determines the unstable component by working backward from future shocks. Each component generally combines household values and distributions: the separation incorporates the feedback through equilibrium prices. Their initial combination must match the inherited distribution and any other predetermined states.

Theorem 2 turns this matching problem into a joint determinacy criterion and solution formula. Every admissible response is the sum of a particular response to shocks and a stable adjustment. Existence requires that the remaining stable adjustment can match the inherited data. Uniqueness requires that only one adjustment can do so. Once it is selected, the same formula gives the entire equilibrium path and its impulse responses. The map Γ_s from stable initial adjustments to predetermined data records both conditions. Its range identifies compatible inherited-data mismatches after accounting for shocks, and its kernel identifies indeterminacy. This approach can be regarded as a function-space extension to the famous Blanchard–Kahn conditions. Under exponential separation and a closed-range condition, this result applies to differential operators on Hilbert spaces as well as to their finite spatial realizations.

The central constructive result, Proposition 1, makes this mechanism operational by building the equilibrium decomposition from household responses. For the household block with finitely many aggregate price inputs, we begin with a known split at fixed prices. Market clearing then adds feedback: changes in saving redistribute households, redistribution changes prices, and those prices alter household values and saving. The proposition reconstructs the

correction to the fixed-price response at each frequency using the household response factors underlying fake news and small price-consistency solves. Adding its frequency integral to the known household stable projection yields the equilibrium stable projection. Its complement identifies the unstable component. These projections can be applied to candidate household profiles without forming a full equilibrium eigendecomposition or storing a basis for its stable space. The household factors reconstruct value and distribution responses, retaining the information needed to impose economic initial and terminal conditions.

The same mechanism supplies exact continuation for an SSJ calculation. At a matching date T within a stationary tail, future shocks determine the admissible unstable component, while the stable component matches the state reached by the finite prefix. Theorem 3 shows that imposing this condition is equivalent to solving the original infinite-horizon problem. The continuation condition enters the household Jacobian calculation itself, so anticipated future prices affect decisions within the prefix. For a stationary steady-state linearization, T is only a computational interface. Exact prefix and continuation calculations introduce no horizon truncation error. Spatial discretization, time discretization, and numerical projection errors remain separate.

The geometric formulation supports this solution procedure by keeping household optimality, distributional transport, market clearing, and boundary data in one fiber product equilibrium paths relation. It can be transformed into a forward-backward representation and an aggregate SSJ closure representation. The former retains the forward-backward dynamics by solving static equations. It identifies which existence and uniqueness restrictions survive a valid elimination of internal household paths. The aggregate SSJ closure retains only aggregate paths by eliminating internally determined household paths. This makes it straightforward to compute the impulse responses of aggregate data. The continuous-state results state the compatible boundary domains and response estimates explicitly. For numerical computation, we use the discrete exterior calculus (DEC) scheme developed in our companion paper Du (2026), based on compatible discrete differential operators (Desbrun et al., 2005). Its constant-preserving backward dynamics, mass-preserving forward dynamics, and mass-zero policy derivatives preserve the distributional restrictions needed by the decomposition. A calibrated Aiyagari application implements the common determinacy and impulse-response calculation. We also compare finite-prefix SSJ solutions across time grids and matching dates.

Relation to the literature. The SSJ approach of Auclert et al. (2021) provides the path-space computational foundation. Bilal and Goyal (2026) derive analytical household policy responses in continuous time and use variable time grids. Those responses can supply the household factors in our construction. Building on the continuous-time HJB–KFE setting of Achdou et al. (2022), we use these factors to construct the equilibrium decomposition and its initial and terminal restrictions. Auclert et al. (2025) study determinacy and solution through the quasi-Toeplitz structure of stationary discrete-time Jacobians, linking operator theory, compact corrections, and iterative computation. Glawion (2026) provides, to our knowledge, the first continuous-time extension of this determinacy and large-scale solution framework, using Wiener–Hopf operators with compact boundary corrections. Our route constructs the required

stable and unstable projections from the underlying household dynamics and price feedback, then uses them to assess determinacy and solve for responses. We extend the stable–unstable saddle-path decomposition underlying Blanchard and Kahn (1980) from finite-dimensional systems to function spaces of household values and distributions. Under the stated dichotomy assumptions, bijectivity of the stable-trace map Γ_s gives the function-space analogue of their count and rank conditions: every admissible inherited state selects exactly one stable initial adjustment. The dichotomy argument draws on established half-line boundary-value theory. Morris (2011) gives a related nonlinear Fredholm treatment. Our constructive contribution connects that boundary-value mechanism to household response operators and reuses the resulting decomposition for exact continuation within SSJ. With valid internal elimination, aggregate price or quantity paths remain the unknowns of the SSJ solve.

Section 2 connects the Blanchard–Kahn count and rank conditions to the matching of household functions and motivates the construction of the stable filter. Section 3 prepares the household operators, admissible paths, and equilibrium geometry. Section 4 establishes the existence and uniqueness criteria and constructs the stable and unstable subspaces from household responses. Section 5 proves equivalence with the aggregate SSJ closure and uses the projections for initial-data matching and exact continuation. Section 6 implements the method in Aiyagari, computes impulse responses, and compares finite-prefix SSJ calculations across time grids and matching dates. Section 7 concludes. Appendices provide the adjoint compatibility analysis, frequency and index comparisons, proofs, and computational derivations.

2 From Blanchard–Kahn conditions to geometry of heterogeneity

The Blanchard–Kahn solution matches predetermined variables with stable combinations of predetermined and forward-looking variables. Our method asks the same economic question when the inherited state is a household distribution and the forward-looking object is a value function. The central issue is how to make this match when there is no finite list of coordinates to count. We first identify the matching operation behind the usual count and rank conditions, then explain how household responses supply the stable family needed to carry out this match for functions.

2.1 What the Blanchard–Kahn count checks

Consider a continuous-time illustration of the saddle-path mechanism in Blanchard and Kahn (1980). Let x be an inherited productive-asset or income-generating state, and let y be a forward-looking valuation that can jump. Both are deviations from steady state:

$$\dot{x} = -\alpha x + \beta y, \quad \dot{y} = \rho y - \kappa x, \quad x(0) = x_0. \quad (2.1)$$

The state depreciates at rate α , while a higher valuation encourages accumulation through βy . The valuation earns the required return ρ through payouts κx and capital gains. With $\alpha, \rho, \kappa >$

0, $\beta \geq 0$, and $\beta\kappa < \alpha\rho$, there is one negative root λ_s and one positive root. Convergence to steady state selects

$$y_0 = qx_0, \quad q = \frac{\kappa}{\rho - \lambda_s}, \quad x(t) = x_0 e^{\lambda_s t}, \quad y(t) = qx_0 e^{\lambda_s t}. \quad (2.2)$$

An alternative initial valuation adds the growing component. There is one stable direction for one predetermined variable, and the stable line $y = qx$ selects exactly one valuation for each inherited x_0 . When $\beta = 0$, its slope reduces to the present-value coefficient $\kappa/(\rho + \alpha)$. Figure 1 gives a schematic illustration of convergence along stable directions and divergence along unstable directions.

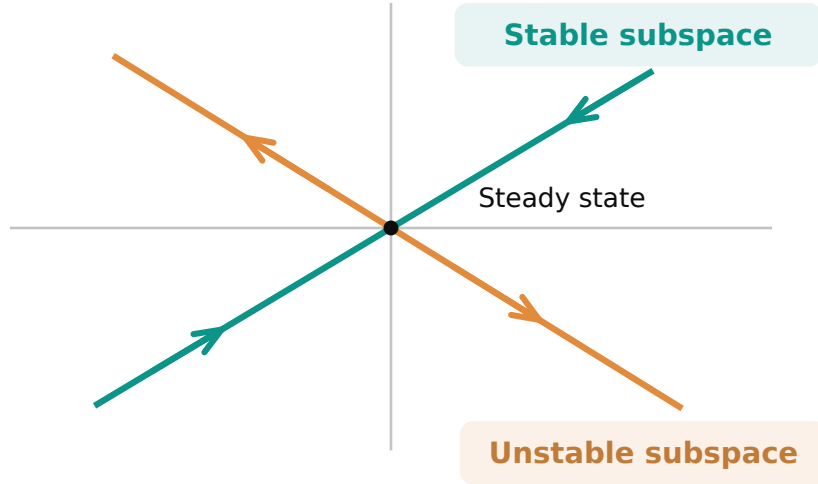


Figure 1: Illustrative saddle-point dynamics. Stable directions converge toward the steady state. Unstable directions diverge in forward time unless the equilibrium boundary conditions select their admissible values.

To see what the count is doing, now let x contain n_x predetermined variables and y contain the forward-looking variables. Suppose the columns of $Q_s = (Q_y^\top, Q_x^\top)^\top$ form a basis for the n_s stable directions. A vector of weights c on those directions produces initial values

$$\begin{pmatrix} y_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} Q_y \\ Q_x \end{pmatrix} c, \quad Q_x c = x_0, \quad y_0 = Q_y Q_x^{-1} x_0 \quad \text{when } Q_x \text{ is invertible.} \quad (2.3)$$

The equilibrium problem is to choose the weights that reproduce the inherited x_0 . Counting $n_s = n_x$ makes the matching matrix Q_x square. The rank condition makes it invertible. With no roots on the stability boundary, the count also says that there are as many unstable roots as jump variables. Together the count and rank conditions ensure that every inherited configuration receives exactly one stable adjustment, and hence one initial choice of the forward-looking variables. Under full rank, too few stable directions leave some inherited states unmatched, while too many leave free choices. Equal counts can still fail if the directions do not span the inherited coordinates. In the (x, y) plane, a vertical stable line illustrates this failure.

Thus the operative requirement is a one-to-one match, with the count and rank providing a finite-dimensional way to check it. The stable directions are combinations of variables. They need not coincide with the predetermined variables themselves.

2.2 From counting coordinates to matching inherited functions

In the heterogeneous household problem, $g_0(a, i)$ replaces the predetermined vector x_0 , and $v_0(a, i)$ replaces the jump vector y_0 . The first is the inherited distribution deviation across assets and income states, with zero total mass. The second is the value-function deviation that must adjust to support the equilibrium transition. A fixed grid gives large vectors and permits a finite-dimensional BK calculation. The continuous problem retains the entire profiles instead.

Under the stability assumptions stated later, let E_s collect the initial value–distribution pairs that generate convergent equilibrium paths without shocks. Reading the inherited component of a pair gives

$$\Gamma_s(v, g) = g, \quad \Gamma_s = \Gamma|_{E_s}. \quad (2.4)$$

The stable-trace map Γ_s performs exactly the operation that Q_x performs in (2.3). It reads which inherited state a stable adjustment can accommodate. The word *trace* here refers only to the initial value of a path. We seek a bijective match: every admissible inherited distribution has a partner in E_s , and no distribution has two different partners. The first requirement gives existence and the second gives uniqueness.

Bijectivity is therefore the function-space counterpart of the BK count *together with* its rank condition. It preserves the economic meaning of counting by comparing the two families through one-to-one matching. There must be no inherited distribution left out and no unused freedom in its stable adjustment. The matching must be the one supplied by the economy through Γ_s . Small changes in the inherited distribution should produce controlled changes in the selected value profile. The function-space assumptions stated later make this requirement precise.

When the match is bijective, it supplies the counterpart of the BK policy rule:

$$\Gamma_s^{-1}g_0 = (\mathcal{F}g_0, g_0), \quad v_0 = \mathcal{F}g_0, \quad \mathcal{F} = \pi_v\Gamma_s^{-1}. \quad (2.5)$$

First find the stable pair that matches g_0 , then read off its value component using π_v . Thus $\mathcal{F}$ plays the role of $Q_yQ_x^{-1}$, or q in the scalar example. It is an equilibrium valuation rule. Household optimality converts the selected values into consumption and saving policies.

2.3 Constructing the matching problem with a stable filter

The matching condition is simple to state, but its domain E_s is an equilibrium object that must be found. With a small matrix, stable eigenvectors or a Schur decomposition provide the required directions. For household functions, the difficulty goes beyond the size of the matrix. A differential operator need not have a complete list of eigenvectors from which to assemble every stable profile. Even individually decaying modes may approach zero decay rates as finer distributional variation is admitted, so stable and unstable behavior need not remain uniformly separated. Finite-grid root counts alone do not establish the continuous result. The separation and response bounds imposed later address these issues.

Our construction represents the stable family by a projection filter $\mathcal{P}_s$, rather than by listing its directions one by one. Applied to a candidate value–distribution pair, the filter retains the

stable component. $\mathcal{P}_u = I - \mathcal{P}_s$ retains the unstable component. This makes it possible to work with the whole family through actions on profiles, without storing a full equilibrium eigenbasis. The constructive contribution in Proposition 1 is to obtain these actions from household responses and aggregate price feedback.

The inherited-state map itself just reads the distribution. To find its inverse, hold g_0 fixed and solve

$$\mathcal{P}_u(v_0, g_0) = 0. \quad (2.6)$$

This is the function-space version of choosing stable weights to satisfy $Q_x c = x_0$. If every inherited g_0 has exactly one solution, Γ_s is bijective and the solution is the valuation rule $v_0 = \mathcal{F}g_0$. A failure to match some profile means nonexistence for that inherited state. More than one match means indeterminacy. Proposition 1 constructs the equilibrium stable filter, while Theorem 2 identifies one-to-one and onto matching as the existence and uniqueness condition. On a numerical grid this returns the familiar count and rank checks. In function spaces it states the corresponding requirement without relying on a finite basis. Once the initial match is selected, the same split is used later to compute forced transitions and impulse responses.

3 Household operators and equilibrium geometry

The equilibrium unknowns are paths of prices and quantities. To describe their economic restrictions, we retain the household value and distribution as internal paths. Prices affect decisions, decisions move households across individual states, and aggregation feeds those movements back into market clearing. This section defines the operators in that loop and the admissible path spaces on which the equilibrium derivative $\mathcal{D}_{\text{eq}}$ acts. Section 4 uses their closed equilibrium realization to study stable traces and construct the stable–unstable split. Section 5 relates this representation to the aggregate SSJ closure and imposes its initial and terminal restrictions. Section 6 implements the resulting equilibrium and SSJ calculations.

3.1 Levels, steady states, and first-order responses

A variable without a perturbation marker denotes a level in the nonlinear economy. Thus $V(t, a, i)$ is the value function, $\mu(t)$ the household distribution, $p(t)$ a vector of prices, and $z(t)$ a vector of aggregate exogenous variables. A bar denotes the stationary reference equilibrium. Consider an experiment indexed by its size ε , such as a small productivity disturbance. We write

$$\begin{aligned} V^\varepsilon &= \bar{V} + \varepsilon v + o(\varepsilon), & \mu^\varepsilon &= \bar{\mu} + \varepsilon g + o(\varepsilon), \\ p^\varepsilon &= \bar{p} + \varepsilon \delta p + o(\varepsilon), & z^\varepsilon &= \bar{z} + \varepsilon \delta z + o(\varepsilon). \end{aligned} \quad (3.1)$$

Here $v = \delta V$ and $g = \delta \mu$ are convenient shortcuts for first-order deviations. Lower case by itself does not mean a deviation. For example, p is a level and δp its derivative with respect to ε . The actual price change is approximately $\varepsilon \delta p$. All deviations are in levels unless a log deviation is explicitly specified. A percentage response is obtained by dividing by the relevant steady-state level. A signed distribution change g is not a probability distribution.

The perturbation derivative δ should also be distinguished from a time derivative ∂_t , a wealth derivative ∂_a , and an operator derivative $D\mathcal{A}$. In particular, $D\mathcal{A}(\bar{p})[\delta p]$ means the first-order change in an entire output path when the entire input price path changes in direction δp . A hat is reserved for the Laplace transform discussed in Appendix C.5. It does not denote a first-order deviation. We use z_i for an idiosyncratic productivity state in the Aiyagari example. that state is distinct from the vector of aggregate shocks $Z(t)$.

Table 1: Notation for the economic objects and their linear responses

Economic object	Level	First-order response	Interpretation
Value function	V , steady state $\bar{V}$	$v = \delta V$	Change in household continuation value
Distribution	μ , steady state $\bar{\mu}$	$g = \delta \mu$	Redistribution of a fixed population
Prices	p , steady state $\bar{p}$	δp	For example, $(\delta r, \delta w)$
Consumption rule	c , steady state $\bar{c}$	δc	Response at a given individual state
Aggregate output	Y , steady state $\bar{Y}$	δY	Response after aggregation
Exogenous inputs	z , steady state $\bar{z}$	δz	Specified shock paths

3.2 Path spaces and linear response operators

An SSJ takes a small input path and returns the associated small output path. For a household aggregate map $Y = \mathcal{A}(p)$, its derivative is

$$\delta Y = \mathcal{J} \delta p, \quad \mathcal{J} = D\mathcal{A}(\bar{p}), \quad \delta Y(t) = \int_0^\infty J(t, \tau) \delta p(\tau) d\tau, \quad (3.2)$$

with a direct multiplication term included when necessary. The two dates have different roles: τ is the date of the price change, and t is the date at which we measure the aggregate response. Anticipation permits $t < \tau$. A kernel column for a dated impulse is a convenient distributional notation. Actual admissible inputs can be ordinary square-integrable paths or smooth approximations to that impulse.

We need a class of paths in which “returning to equilibrium” has a precise meaning. Our baseline is the space of square-integrable responses. For a vector or function-valued path x , define

$$L_\eta^2(E) = \left\{ x : \int_0^\infty e^{-2\eta t} \|x(t)\|_E^2 dt < \infty \right\}, \quad \|x\|_\eta^2 = \int_0^\infty e^{-2\eta t} \|x(t)\|_E^2 dt. \quad (3.3)$$

The baseline is $\eta = 0$. For aggregate variables, E can simply be Euclidean space with a chosen scaling of its components. For values or distributions, the norm also aggregates across individual states. A Hilbert space is a setting in which these norms arise from an inner product, so orthogonality and least-squares projections are available. These technical choices make the kernel and cokernel calculations in Section 4 well defined.

A positive η allows some growing responses and changes admissibility. Permanent shocks are better handled by specifying the new steady state or the relevant center dynamics explicitly. A strong formulation controls $\dot{x}$ separately: H^1 requires both x and $\dot{x}$ to be square integrable, and a PDE graph norm also controls the spatial operator. Section 4.2 instead admits mild solutions of unbounded differential operators. Its graph norm controls the path, the dynamic residual, and the initial trace without requiring the two derivatives separately. The initial trace is the well-defined value $x(0)$ in either formulation. The chosen realization must retain both the dynamic equations and the inherited distribution.

3.3 Household decisions and the evolution of the distribution

Let Ω be the individual state space. The backward Markov generator $\mathcal{L}_{\alpha,p}$ describes how a function of individual states changes when households follow policy α at prices p . A generator is the continuous-time counterpart of a transition matrix. If $\varphi(a, i)$ is an observable, such as assets or consumption, $e^{t\bar{\mathcal{L}}}\varphi$ is its expected value after t units of time, conditional on the current state and steady-state behavior. The forward operator $e^{t\bar{\mathcal{L}}^*}$ instead evolves a distribution.

These are two descriptions of the same household transitions. The pairing is the aggregate

$$\langle \varphi, \mu \rangle = \sum_i \int \varphi(a, i) \mu_i(da).$$

Computing that aggregate by moving households forward or by moving the observable backward must give the same answer:

$$\langle e^{t\bar{\mathcal{L}}}\varphi, \mu \rangle = \langle \varphi, e^{t\bar{\mathcal{L}}^*}\mu \rangle, \quad \langle \mathcal{L}\varphi, \mu \rangle = \langle \varphi, \mathcal{L}^*\mu \rangle, \quad \mathcal{L}\mathbf{1} = 0. \quad (3.4)$$

This is the economic meaning of adjointness. The last identity says that expected population size does not change. Boundary conditions are included in the generator. Borrowing constraints or boundary masses must be treated with the corresponding weak flux conditions.

In levels, household optimality, distribution dynamics, and market clearing take the form

$$\begin{aligned} 0 &= \rho V - \partial_t V - \mathcal{H}(V, p, z), \\ 0 &= \partial_t \mu - \mathcal{L}_{\alpha(V,p,z),p,z}^* \mu, \\ 0 &= \mathcal{C}(V, \mu, p, z), \quad \mu(0) = \mu_0. \end{aligned} \quad (3.5)$$

The optimized Hamiltonian $\mathcal{H}$ collects current utility and expected continuation value. $\mathcal{C}$ collects equilibrium restrictions. The HJB determines optimal behavior given prices, while the KFE determines the distribution given behavior. Their joint solution becomes a general equilibrium only when $\mathcal{C} = 0$. Predetermined aggregate states can be added to the distributional state. The formulation concerns perfect-foresight transitions and their first-order impulse responses, not a nonlinear common-noise master equation.

Because total population is normalized to one, $\langle \mathbf{1}, \mu^\varepsilon \rangle = 1$ for every admissible experiment. Differentiating this accounting identity gives

$$\mathcal{M}_0 = \{g : \langle \mathbf{1}, g \rangle = 0\}. \quad (3.6)$$

The space $\mathcal{M}_0$ is the distributional tangent space: a first-order response can move households between states but cannot create households. It is an economic accounting restriction, not an additional equilibrium condition. If the reference distribution is on the boundary of the set of feasible distributions, nonlinear perturbations can be constrained to a cone. Local nonlinear results then require a suitable differentiable chart or constrained argument.

Differentiating the KFE separates the persistence of an existing distribution change from the effect of changed behavior:

$$\dot{g} = \bar{\mathcal{L}}^* g + (D\mathcal{L}[\delta\alpha, \delta p, \delta z])^* \bar{\mu}. \quad (3.7)$$

The first term propagates households using steady-state transitions. The second term real-locates the steady-state population because saving, adjustment, or transition policies have changed. In a regular one-asset density representation, its saving component is $-\partial_a(\bar{\mu} \delta s)$. This is the divergence of the additional asset flow. Its integral is zero when the full boundary flux is accounted for. The same conclusion follows directly by differentiating $\mathcal{L}\mathbf{1} = 0$. Removing the mass direction is necessary for distributional response calculations, but a spectral gap on $\mathcal{M}_0$ still requires a separate mixing assumption.

Linearized household operators. Fix the steady state and temporarily take aggregate prices as exogenous. The analytically differentiated household block can be organized as

$$\begin{aligned} -\dot{v} &= \mathcal{L}_v v + \mathbf{B}_{vp} \delta p, \\ \dot{g} &= \mathcal{L}_g g + \mathbf{B}_{gv} v + \mathbf{B}_{gp} \delta p, \quad g(0) = 0, \\ \delta Y &= \mathcal{O}_g g + \mathcal{O}_v v + \mathcal{O}_p \delta p. \end{aligned} \quad (3.8)$$

Here all lowercase paths on the left are deviations as defined above. The coefficients are evaluated at the steady state. In each coupling map $\mathbf{B}_{ab}$, the first index identifies the affected equation or policy and the second identifies its input. The symbols $\mathcal{O}_g$, $\mathcal{O}_v$, and $\mathcal{O}_p$ denote aggregation from the corresponding inputs. The operator $\mathcal{L}_g = \bar{\mathcal{L}}^*|_{\mathcal{M}_0}$ propagates distribution changes. In the simplest envelope linearization, $\mathcal{L}_v = \bar{\mathcal{L}} - \rho I$ propagates discounted value news. The map $\mathbf{B}_{vp}$ converts a price change into a change in the HJB payoff. The map $\mathbf{B}_{gv}$ translates the resulting policy response into a distributional source. $\mathbf{B}_{gp}$ captures direct effects of prices on individual-state transitions. Both have mass-zero outputs.

The last equation is aggregation. For asset supply, $\mathcal{O}_g g$ is the change in assets due to redistribution and usually $\mathcal{O}_v = \mathcal{O}_p = 0$. For consumption, there is also a response of consumption at a fixed household state: $\mathcal{O}_v v$ captures the value-induced policy effect and $\mathcal{O}_p \delta p$ any direct effect. The aggregation maps apply to the observable under study. Aggregate capital in the Aiyagari example is denoted by $K(t)$.

This representation accommodates analytical policy differentiation of the type developed by Bilal and Goyal (2026). For example, differentiating the consumption policy on a fixed binding set gives the linear map $\delta c = \mathbf{B}_{cv} v + \mathbf{B}_{cp} \delta p$. Here $\mathbf{B}_{cv}$ maps a value-function perturbation into a consumption response, while $\mathbf{B}_{cp}$ maps current price changes into consumption holding the value-function input fixed. In the baseline model, the interior first-order condition $U'(c) = \partial_a V$ gives $\mathbf{B}_{cv} v = \partial_a v / U''(\bar{c})$ and $\mathbf{B}_{cp} = 0$. Prices still affect interior consumption through v . At

a borrowing boundary that remains binding, consumption instead tracks current disposable income. Appendix B.3.4 states the corresponding policy derivatives.

These policy maps determine the distributional maps $\mathbf{B}_{gv}$ and $\mathbf{B}_{gp}$. Writing $b\delta p = a\delta r + y_i\delta w$ for the income response at a fixed household state, the saving response is $\delta s = -\mathbf{B}_{cv}v + (b - \mathbf{B}_{cp})\delta p$. The transport-source map $\mathcal{S}\chi = -\partial_a(\bar{\mu}\chi)$ converts an admissible saving-drift perturbation χ into the instantaneous redistribution of the steady-state population. Thus $\mathbf{B}_{gv} = -\mathcal{S}\mathbf{B}_{cv}$ and $\mathbf{B}_{gp} = \mathcal{S}(b - \mathbf{B}_{cp})$: the consumption response enters saving with a minus sign, and the direct saving response is income minus consumption. Unlike $\mathbf{B}_{cv}$ and $\mathbf{B}_{cp}$, whose outputs are individual policy responses, $\mathbf{B}_{gv}$ and $\mathbf{B}_{gp}$ produce mass-zero sources for the distribution equation. Appendix B.3.5 defines $\mathcal{S}$ weakly, explains its flux interpretation, and derives these identities with the boundary included.

Instead of repeatedly solving the nonlinear household problem for perturbed price sequences, one solves its linear response equation. Under exponential stability of $\mathcal{L}_v$ and $\mathcal{L}_g$, the resulting paths are equivalent to the analytical result in Bilal and Goyal (2026)

$$\begin{aligned} v(t) &= \int_t^\infty e^{\mathcal{L}_v(\tau-t)} \mathbf{B}_{vp} \delta p(\tau) d\tau, \\ g(t) &= \int_0^t e^{\mathcal{L}_g(t-r)} \{ \mathbf{B}_{gv} v(r) + \mathbf{B}_{gp} \delta p(r) \} dr. \end{aligned} \tag{3.9}$$

The first line integrates anticipated future price effects backward to today's value. The second integrates past saving responses forward to today's distribution. These are variation-of-constants formulas, often called Duhamel formulas. In the compatible realizations considered here, the coupling maps are bounded between the selected function spaces. Appendix C.8 gives sufficient weak-flux conditions for this property. Unbounded boundary inputs require a separately justified admissibility formulation.

We denote these backward and forward solution operations by $\mathcal{V}$ and $\mathcal{G}$: $v = \mathcal{V}\delta p$ and $g = \mathcal{G}(\mathbf{B}_{gv}v + \mathbf{B}_{gp}\delta p)$ for zero initial distribution deviation. A prescribed g_0 adds $e^{\mathcal{L}_g t}g_0$ to the forward response. The maps $\mathcal{L}_v, \mathcal{L}_g, \mathbf{B}_{vp}, \mathbf{B}_{gv}, \mathbf{B}_{gp}$, together with the aggregation maps $\mathcal{O}_g, \mathcal{O}_v, \mathcal{O}_p$, are the building blocks of the full derivative $\mathcal{D}_{\text{eq}}$. Appendix D.1 composes them into aggregate Jacobians. Their local evolution equations also give a realization of those path operators through which the stable-subspace and continuation calculations can be made.

3.4 Closing the household block with the rest of the economy

The household block describes decisions and distributional responses given prices. General equilibrium requires those same prices to be consistent with firms, fiscal and monetary policy, and market clearing. In a one-asset production economy, a trial capital path determines factor prices. Households respond to those prices. Their asset holdings must reproduce the original capital path. This feedback is what we call *closure*. Geometry gives a precise description of the intersection of these mutually consistent sectoral choices.

For sector i , let $\mathcal{S}_i(z)$ collect all paths satisfying that sector's equations and boundary conditions for a given exogenous path. The interface map $\pi_i : \mathcal{S}_i(z) \rightarrow \mathcal{I}_i$ extracts the prices and quantities that the sector exchanges with others. $\mathcal{I}_i$ is the space of these shared paths.

Let the market-consistency relation $\mathcal{C}_{\text{mkt}}(z) \subseteq \prod_i \mathcal{I}_i$ impose agreement on those interfaces: the same wage is paid and received, assets supplied equal assets demanded, and the relevant budgets balance. The complete equilibrium relation is

$$\mathcal{E}_{\text{GE}}(z) = \left(\prod_i \mathcal{S}_i(z) \right) \times_{\prod_i \mathcal{I}_i} \mathcal{C}_{\text{mkt}}(z). \quad (3.10)$$

The term *fiber product* means that we retain precisely the sectoral solutions that agree on their shared economic variables. It does not introduce a new equilibrium condition. Its advantage is to define the complete equilibrium before choosing which prices or aggregates a numerical solver will treat as unknowns. It is a smooth manifold only under appropriate regularity conditions.

For example, solving for the capital path or for the interest-rate path can be two parameterizations of the same equilibrium, provided the conversion is locally invertible. An evaluation graph for partial-equilibrium blocks depends on that choice. The underlying household–firm consistency relation does not. This distinction is useful when both households and firms are heterogeneous and their distributions feed back into each other’s decisions.

Let x collect the endogenous *level* paths and let $\mathcal{F}_{\text{eq}}(x, z)$ stack their equilibrium residuals. Its first-order approximation is

$$\begin{aligned} \mathcal{F}_{\text{eq}}(x, z) &= 0, & \mathcal{D}_{\text{eq}} &= D_x \mathcal{F}_{\text{eq}}(\bar{x}, \bar{z}) : \mathcal{X} \rightarrow \mathcal{R}_{\text{eq}}, \\ \mathcal{D}_{\text{eq}} \delta x &= \mathcal{B}_{\text{eq}, z} \delta z, & \mathcal{B}_{\text{eq}, z} &= -D_z \mathcal{F}_{\text{eq}}(\bar{x}, \bar{z}). \end{aligned} \quad (3.11)$$

The map $\mathcal{B}_{\text{eq}, z}$ translates exogenous changes into forcing of the equilibrium equations. Its subscripts identify the affected equation block and the shock input. The residual space $\mathcal{R}_{\text{eq}}$ includes the predetermined initial data as well as the dynamic and market-clearing equations. Known redundancies, such as one Walras-law equation, must be removed before counting existence restrictions. The linearized household operators in (3.8) supply its optimality and distribution blocks.

Solving static equations of market clearing in $\mathcal{D}_{\text{eq}}$ produces a forward-backward representation. The saddle-path selection is conveniently operated with this representation in Section 4.2. Eliminating internally determined household paths gives an aggregate SSJ closure representation. Section 5.1 proves that valid elimination preserves the uniqueness and existence tests. These are representations of the same linearized fiber-product equilibrium, preserving its existence and uniqueness conditions. The fiber product therefore explains which sectoral responses must fit together. The SSJ Jacobians quantify their responses. And our projection construction studies the admissible dynamics of the resulting coupled equilibrium.

Changing coordinates gives $\tilde{\mathcal{D}} = U \mathcal{D} V$, with invertible transformations U, V . Although the matrix entries change, the set of homogeneous equilibrium responses and the number of independent compatibility restrictions do not. This is the precise sense in which the closure geometry is independent of the chosen economic coordinates.

An evaluation graph orders the sectoral calculations conditional on proposed aggregate paths. Its nodes may include household decisions, distributional transport, firms, and policy rules. Market clearing closes that graph by requiring agreement on the interfaces. The aggregate unknowns are a locally independent set of paths sufficient to specify those interfaces. Changing

that set changes the evaluation graph and the numerical representation, while valid elimination preserves the complete equilibrium relation. Determinacy therefore concerns $\mathcal{D}_{\text{eq}}$, including all boundary data, rather than a particular ordering of partial-equilibrium calculations.

4 Determinacy and the forward-backward representation

Section 3 defines the household operators and their equilibrium coupling. We now ask whether all sectors can agree on one response to the same shock. Two separate economic questions arise. *Uniqueness* asks whether a second admissible response could satisfy the same equations with the same inherited states and shocks. *Existence* asks whether those inherited states and shocks can be accommodated by any admissible response. The first concerns movements the equilibrium equations do not pin down. The second concerns restrictions endogenous adjustment cannot satisfy. The kernel and cokernel are the linear-algebraic objects that measure these two failures.

We first state the general path-space criterion, then express it through stable equilibrium dynamics and construct the required stable and unstable subspaces from household responses. Section 5 connects this representation to the aggregate sequence-space closure and exact continuation. Throughout, $u(t)$ denotes a stacked vector of first-order deviations. It is not a vector of level variables.

4.1 The path operator, its kernel, and its obstruction space

Let $\delta x = (v, g, \delta p)$ collect the value, distribution, and price responses. Stack the linearized HJB, KFE, and market-clearing residuals in that order:

$$\mathcal{D}_{\text{eq}} \begin{pmatrix} v \\ g \\ \delta p \end{pmatrix} = \begin{pmatrix} -\partial_t - \mathcal{L}_v & -\mathbf{B}_{vg} & -\mathbf{B}_{vp} \\ -\mathbf{B}_{gv} & \partial_t - \mathcal{L}_g & -\mathbf{B}_{gp} \\ \mathcal{C}_v & \mathcal{C}_g & \mathcal{C}_p \end{pmatrix} \begin{pmatrix} v \\ g \\ \delta p \end{pmatrix}, \quad (4.1)$$

The initial-condition equations are appended to this displayed interior operator. The domain includes the relevant spatial boundary conditions and admissibility at infinity. Here $\mathbf{B}_{vp}$ is the effect of prices on the household optimality equation, $\mathbf{B}_{gv}$ the policy-induced redistribution map, and $\mathbf{B}_{gp}$ the direct price-induced redistribution map. In each coupling map $\mathbf{B}_{ab}$, the first subscript identifies the affected equation and the second the input variable. In the basic household problem $\mathbf{B}_{vg} = 0$, an atomistic household takes prices as given. So the distribution affects it through prices. Richer specifications can produce a direct distribution dependence.

The last-row maps differentiate the nonlinear equilibrium residual $\mathcal{C}(V, \mu, p, z)$ in (3.5):

$$\mathcal{C}_v = D_V \mathcal{C}, \quad \mathcal{C}_g = D_\mu \mathcal{C}, \quad \mathcal{C}_p = D_p \mathcal{C},$$

with all derivatives evaluated at $(\bar{V}, \bar{\mu}, \bar{p}, \bar{z})$. Their subscripts identify the perturbation inputs $v = \delta V$, $g = \delta \mu$, and δp . Economically, $\mathcal{C}_v v$ measures how changed household values and the associated policy responses affect market clearing. $\mathcal{C}_g g$ measures the effect of redistribution on aggregate supply and demand. $\mathcal{C}_p \delta p$ measures the effect of changing prices while holding the other inputs fixed. These maps produce equilibrium residuals. The symbol $\mathcal{C}_{\text{mkt}}$ in (3.10)

denotes the market-consistency relation, whereas $\mathcal{C}$ is a residual function whose zero condition imposes equilibrium restrictions.

For an exogenous perturbation δz , the market-row forcing is $b_p = -D_z \mathcal{C} \delta z$, evaluated at the same steady state. The equation $\mathcal{C}_v v + \mathcal{C}_g g + \mathcal{C}_p \delta p = b_p$ requires prices and endogenous household responses to offset the imbalance created by the disturbance. When $\mathcal{C}_p$ is boundedly invertible, (4.5a) solves this restriction for the price response.

To say that δx lies in the kernel is to set all these residuals, all shock inputs, and all initial-data deviations to zero and still obtain a nonzero path. Adding that path to an existing response leaves the economic experiment unchanged. Conversely, a left-null direction is a set of weights on the equations whose weighted residual cannot be changed by any endogenous path. If the exogenous shock generates a nonzero residual in that direction, the equilibrium equations are incompatible. These are the infinite-path analogues of the right and left nullspaces of a matrix.

Assumption 1 (Regular linear closure). *The chosen path and residual spaces $\mathcal{X}, \mathcal{R}_{\text{eq}}$ are Hilbert spaces. The differential expressions in (4.1), including their traces, define a bounded map $\mathcal{D}_{\text{eq}} : \mathcal{X} \rightarrow \mathcal{R}_{\text{eq}}$ when $\mathcal{X}$ carries its graph norm. The range is closed. The forcing map $\mathcal{B}_{\text{eq},z} : \mathcal{Z} \rightarrow \mathcal{R}_{\text{eq}}$ is bounded. Independent normalization and accounting identities have been dealt with in defining these spaces.*

Definition 1 (Closure defects and admissible disturbances). *The linearized indeterminacy space and obstruction space are*

$$\mathcal{H}_{\text{eq}}^0 = \ker \mathcal{D}_{\text{eq}}, \quad \mathcal{H}_{\text{eq}}^1 = \mathcal{R}_{\text{eq}} / \text{Ran } \mathcal{D}_{\text{eq}} \simeq \ker \mathcal{D}_{\text{eq}}^*. \quad (4.2)$$

The economically admissible disturbance set is

$$\mathcal{Z}_{\text{adm}} = \{\delta z \in \mathcal{Z} : \mathcal{B}_{\text{eq},z} \delta z \in \text{Ran } \mathcal{D}_{\text{eq}}\}. \quad (4.3)$$

The adjoint in this definition is the Hilbert-space adjoint of the complete operator, including the boundary component. $\text{Ran } \mathcal{D}_{\text{eq}} = \{\mathcal{D}_{\text{eq}} \delta x : \delta x \in X\}$ is the set of sequence space data that some admissible response path can satisfy, where δx collects the entire paths of value-function, distribution, and price responses. The space X imposes the required spatial boundary conditions and infinite-horizon admissibility. $\mathcal{B}_{\text{eq},z} \delta z$ specifies what exogenous disturbance requires the equations to accommodate. $\mathcal{B}_{\text{eq},z} \delta z \in \text{Ran } \mathcal{D}_{\text{eq}}$ means that at least one admissible response δx solves the equilibrium equations. It asks whether adjustments in household choices, the distribution, and prices can reconcile the disturbance with all equilibrium conditions.

Theorem 1 (Linear geometric closure). *Under Assumption 1, the following statements hold.*

1. *For a given δz , an equilibrium perturbation exists if and only if $\langle \psi, \mathcal{B}_{\text{eq},z} \delta z \rangle_{\mathcal{R}_{\text{eq}}} = 0$ for every $\psi \in \ker \mathcal{D}_{\text{eq}}^*$.*
2. *Whenever a solution x_p exists, the full solution set is $x_p + \mathcal{H}_{\text{eq}}^0$. Hence uniqueness of existing solutions is equivalent to $\mathcal{H}_{\text{eq}}^0 = \{0\}$.*

3. *There is a unique continuously dependent solution for every residual forcing in $\mathcal{R}_{\text{eq}}$ if and only if both defect spaces vanish. In that case $\mathcal{D}_{\text{eq}}^{-1}$ is bounded.*
4. *If $\mathcal{D}_{\text{eq}}$ is Fredholm, its index is $\dim \mathcal{H}_{\text{eq}}^0 - \dim \mathcal{H}_{\text{eq}}^1$. An index of zero does not imply that either defect vanishes.*

Proof. The closed-range theorem gives $\text{Ran } \mathcal{D}_{\text{eq}} = (\ker \mathcal{D}_{\text{eq}}^*)^\perp$, proving the first assertion. Subtracting two solutions proves the second. A bounded bijection between Banach spaces has a bounded inverse, proving the third. The fourth is the definition of the Fredholm index. Equal nonzero kernel and cokernel dimensions demonstrate why zero index is insufficient. $\square$

$\mathcal{H}^0$ counts unpinned equilibrium movements and $\mathcal{H}^1$ counts independent compatibility restrictions. A Fredholm operator is one with closed range and finitely many such directions. Its index reports their difference, so a zero index can conceal equally many failures on each side. Geometrically, the two spaces are associated with the two-term deformation complex

$$0 \longrightarrow \mathcal{X} \xrightarrow{\mathcal{D}_{\text{eq}}} \mathcal{R}_{\text{eq}} \longrightarrow 0. \quad (4.4)$$

Equilibrium closure means exactness at both nonzero terms. This is an analytic statement about economic dynamics and boundary conditions.

4.2 Saddle-path selection and a constructive determinacy criterion

The usual saddle-path logic in macroeconomics provides a constructive way to evaluate the abstract criterion. Predetermined states, such as the household distribution, are inherited at date zero. Jump variables, such as values and some asset prices, adjust to make the subsequent response admissible. Eliminate static variables, the remaining first-order system is

$$\dot{u} = Au + f, \quad \Gamma u(0) = d. \quad (4.5)$$

The state u includes forward-looking functions and mass-zero distribution states. The boundary space $\mathbf{D}$ contains predetermined data, and $\Gamma : E \rightarrow \mathbf{D}$ is bounded.

The operator A is obtained by solving the static equations in $\mathcal{D}_{\text{eq}}$ and rearranging its differential equations. To make the construction explicit, denote the three interior forcing components in (4.1) by (b_v, b_g, b_p) , in HJB, KFE, and market-clearing order. They are the corresponding components of $\mathbf{B}_{\text{eq},z}\delta z$. Inherited initial data are imposed separately. Suppose the instantaneous price block $\mathcal{C}_p$ is boundedly invertible. The market equation can then be solved at each date:

$$\mathcal{C}_v v + \mathcal{C}_g g + \mathcal{C}_p \delta p = b_p, \quad \delta p = -\mathcal{C}_p^{-1} \mathcal{C}_v v - \mathcal{C}_p^{-1} \mathcal{C}_g g + \mathcal{C}_p^{-1} b_p. \quad (4.5a)$$

Substitute this expression into the first two rows of (4.1), and multiply the HJB row by -1 to isolate $\dot{v}$. With $u = (v, g)$, the resulting spatial operator and forcing in (4.5) are

$$A = \begin{pmatrix} -\mathcal{L}_v + \mathbf{B}_{vp}\mathcal{C}_p^{-1}\mathcal{C}_v & -\mathbf{B}_{vg} + \mathbf{B}_{vp}\mathcal{C}_p^{-1}\mathcal{C}_g \\ \mathbf{B}_{gv} - \mathbf{B}_{gp}\mathcal{C}_p^{-1}\mathcal{C}_v & \mathcal{L}_g - \mathbf{B}_{gp}\mathcal{C}_p^{-1}\mathcal{C}_g \end{pmatrix}, \quad (4.5b)$$

$$f = \begin{pmatrix} -b_v - \mathbf{B}_{vp}\mathcal{C}_p^{-1}b_p \\ b_g + \mathbf{B}_{gp}\mathcal{C}_p^{-1}b_p \end{pmatrix}.$$

The products involving $\mathcal{C}_p^{-1}$ are the price feedback implied by market clearing. They transmit changes in values and the distribution back into household optimality and distribution dynamics.

For (3.8), take $\mathbf{B}_{vg} = 0$. Suppose market clearing gives $\delta p = P_g g + P_z \delta z$, where P_g maps redistribution into equilibrium price changes and P_z gives the direct price effect of the exogenous disturbance. Ordinary P denotes price maps. Calligraphic $\mathcal{P}_s$ and $\mathcal{P}_u$ below denote the stable and unstable dynamical projections. Then $\mathcal{C}_v = 0$, $\mathcal{C}_g = -P_g$, and $\mathcal{C}_p = I$. If shocks enter this household block only through prices, the formulas become

$$\mathbf{A} = \begin{pmatrix} -\mathcal{L}_v & -\mathbf{B}_{vp}P_g \\ \mathbf{B}_{gv} & \mathcal{L}_g + \mathbf{B}_{gp}P_g \end{pmatrix}, \quad f = \begin{pmatrix} -\mathbf{B}_{vp}P_z \\ \mathbf{B}_{gp}P_z \end{pmatrix} \delta z. \quad (4.5c)$$

The upper-right block captures how redistribution changes prices and hence household values. The lower-left block converts changed values into saving and redistribution. $\mathbf{B}_{gp}P_g$ adds the direct effect of equilibrium prices on distribution dynamics. In particular, $\mathcal{L}_g$ alone is the household distribution generator at fixed prices, whereas $\mathbf{A}$ governs the coupled equilibrium response. The grid-based Aiyagari system in (6.7) is a concrete realization of this construction.

If the only predetermined variable is the distribution, the inherited condition $g(0) = g_0$ becomes $\Gamma = (0 \ I)$ and $d = g_0$. Other predetermined aggregate states add components to u and rows to Γ . The distribution remains in $\mathcal{M}_0$, and the spatial boundary conditions and infinite-horizon admissibility remain part of the domain. Eliminated prices are reconstructed from (4.5a). Under valid elimination, the homogeneous equilibrium paths therefore correspond to

$$\ker \mathcal{D}_{\text{eq}} \simeq \{u : \dot{u} = \mathbf{A}u, \Gamma u(0) = 0, u \text{ admissible}\}. \quad (4.5d)$$

Thus $\mathbf{A}$ supplies the dynamic coefficients of the reduced path problem. The full residual also contains the time derivative and the boundary restrictions.¹ $\mathbf{A}$ helps uncover the tangent directionis of the equilibrium set: it supplies dynamic evolution, while Γ supplies their initial-data restrictions.

A stable initial direction generates a decaying homogeneous response. An unstable direction generates an explosive one unless offset by the response to future shocks. An *exponential dichotomy* is a stable–unstable separation with a uniform positive distance from zero growth. Its projections $\mathcal{P}_s$ and $\mathcal{P}_u$ select those two components, and E_s denotes all admissible homogeneous initial combinations. The word *trace* below simply refers to the initial value of a path. Thus the stable-trace map Γ_s asks which predetermined initial values can be generated by nonexplosive equilibrium dynamics.

Assumption 2 (Dichotomy realization on a function space). *Let E and $\mathbf{D}$ be real Hilbert spaces. The operator $\mathbf{A} : D(\mathbf{A}) \subset E \rightarrow E$ is closed and densely defined. Its domain includes the spatial boundary conditions and accounting restrictions. Complementary bounded projections $\mathcal{P}_s, \mathcal{P}_u$*

¹Assumption 2 permits closed unbounded HJB–KFE operators. Appendix C.3 constructs their Green map and initial trace in a mild graph space. Separate H^1 and spatial domain regularity require additional estimates and a compatible strong trace space. Appendix C.8 distinguishes the mixing supplied by borrowing-floor regeneration from the extra boundary and policy-flux conditions needed for a continuous household realization.

preserve $D(\mathbf{A})$ and commute with $\mathbf{A}$ there. Write $E_j = \text{Ran } \mathcal{P}_j$, $j = s, u$, and let $\mathbf{A}_j$ be the corresponding part of $\mathbf{A}$. The operator $\mathbf{A}_s$ generates a strongly continuous forward semigroup on E_s , and $-\mathbf{A}_u$ generates a strongly continuous backward semigroup on E_u , satisfying

$$\|e^{t\mathbf{A}_s}\mathcal{P}_s\| \leq Me^{-\alpha t}, \quad \|e^{-t\mathbf{A}_u}\mathcal{P}_u\| \leq Me^{-\alpha t}, \quad t \geq 0, \quad \alpha > 0. \quad (4.6)$$

The initial-data map $\Gamma : E \rightarrow \mathbf{D}$ is bounded and $\Gamma_s = \Gamma|_{E_s}$ has closed range. When finite defect dimensions are claimed, Γ_s is assumed Fredholm.

No forward semigroup on all of E is required: the value equation may be a backward differential equation. Below, $e^{t\mathbf{A}}\mathcal{P}_s$ for $t \geq 0$, and $e^{t\mathbf{A}}\mathcal{P}_u$ for $t \leq 0$, denote only these restricted semigroups. Appendix C.8 connects household mixing and weak boundary fluxes to such function-space realizations. Equilibrium feedback still has to satisfy the separation conditions of Proposition 1.

For $f \in L^2(\mathbb{R}_+; E)$, define

$$\begin{aligned} (\mathcal{K}f)(t) &= \int_0^t e^{\mathbf{A}(t-r)}\mathcal{P}_s f(r)dr - \int_t^\infty e^{\mathbf{A}(t-r)}\mathcal{P}_u f(r)dr, \\ r_f &= (\mathcal{K}f)(0) = - \int_0^\infty e^{-\mathbf{A}r}\mathcal{P}_u f(r)dr. \end{aligned} \quad (4.7)$$

The negative-time exponential is used only on the unstable subspace, where it decays. These integrals are bounded maps into the relevant solution and trace spaces. The first integral accumulates past forcing in stable directions. The second works backward from admissibility at infinity to determine today's unstable component from future forcing. Consequently, r_f is the initial adjustment forced by the shock, and the remaining freedom is a stable homogeneous response. This is the forward-looking economic content of the Green operator $\mathcal{K}$.

For an unbounded $\mathbf{A}$, admissibility means a continuous $L^2(\mathbb{R}_+; E)$ *mild solution*: its stable component obeys forward variation of constants and its unstable component obeys backward variation of constants. Neither $\dot{u}$ nor $\mathbf{A}u$ must separately belong to L^2 . Let $\mathcal{X}_\mathbf{A}$ be the space of these paths with an L^2 forcing, equipped with

$$\|u\|_{\mathcal{X}_\mathbf{A}}^2 = \|u\|_{L^2}^2 + \|f\|_{L^2}^2 + \|u(0)\|_E^2.$$

The forcing is unique and defines the mild residual $\dot{u} - \mathbf{A}u$. Appendix C.3 specifies these identities and proves that this is a Hilbert graph space with a bounded initial trace. Comparison with Theorem 1 uses this mild graph realization of the residual, or an additionally justified strong realization with its own trace space.

Theorem 2 (Stable-trace closure criterion). *Under Assumption 2, all admissible mild solutions in $\mathcal{X}_\mathbf{A}$ of (4.5) are*

$$u(t) = e^{t\mathbf{A}}s_0 + (\mathcal{K}f)(t), \quad s_0 \in E_s, \quad \Gamma_s s_0 = d - \Gamma r_f. \quad (4.8)$$

Here s_0 is the initial stable component: since $r_f \in E_u = \text{Ran } \mathcal{P}_u$, one has $u(0) = s_0 + r_f$ and $s_0 = \mathcal{P}_s u(0)$. Thus existence for (f, d) is equivalent to $d - \Gamma r_f \in \text{Ran } \Gamma_s$, and uniqueness is equivalent to $\ker \Gamma_s = \{0\}$. For the bounded boundary-value map $\mathcal{T} : \mathcal{X}_\mathbf{A} \rightarrow L^2(\mathbb{R}_+; E) \times \mathbf{D}$,

$$\mathcal{T}u = (\dot{u} - \mathbf{A}u, \Gamma u(0)), \quad (4.9)$$

the first component is understood as the mild residual. Its range is closed, and its kernel and quotient cokernel are isomorphic to those of Γ_s . If Γ_s is Fredholm, so is $\mathcal{T}$, with the same index. When E_s and $\mathbf{D}$ are finite dimensional, write $n_s = \dim E_s$ and $q = \dim \mathbf{D}$. Then

$$\begin{aligned}\dim \ker \mathcal{T} &= n_s - \text{rank } \Gamma_s, \\ \dim \text{coker } \mathcal{T} &= q - \text{rank } \Gamma_s, \\ \text{ind } \mathcal{T} &= n_s - q.\end{aligned}\tag{4.10}$$

Appendix C.3 proves the theorem using the two restricted semigroups.

The theorem gives a geometric interpretation: the affine admissible-trace set $r_f + E_s$ must meet the affine initial-data set $\{u_0 : \Gamma u_0 = d\}$. A single intersection gives a unique solution, an empty intersection gives nonexistence, and a positive-dimensional intersection gives linear indeterminacy. In finite dimension, having $n_s = q$ is only a dimension count. The map Γ_s must also have full rank: the orientation of the stable space relative to the inherited-data restrictions matters. Appendix C.5 illustrates this distinction and relates it to frequency and index criteria.

The condition $d - \Gamma r_f \in \text{Ran } \Gamma_s$ asks whether the economy can reconcile its inherited state with the announced future shocks. The reference response $\mathcal{K}f$ already satisfies the dynamic equations and the infinite-horizon condition, but its initial predetermined component Γr_f need not equal the actual inherited data d . The remaining admissible adjustment $e^{t\mathbf{A}}s_0$ must supply precisely the difference $d - \Gamma r_f$. The range of Γ_s lists the initial-state corrections that such adjustments can supply. Thus existence means that the required correction is attainable. An unattainable weighted combination of inherited data gives an intertemporal compatibility restriction, analogous to equating inherited government debt with future primary surpluses. Appendix C.4 derives the model-implied discount weights and the corresponding accounting identities.

Example 1 (Income and valuation before and after eliminating a state). *Take the case $\beta = 0$ in Section 2.1: the inherited income state x decays independently of its valuation y . This special case shows how eliminating a predetermined variable changes the stable space and the forcing without changing the economic solution. Use $H^1 \cap L^2$ deviations, the strong form of the mild admissibility condition for this finite-dimensional system. For this calculation only, place income first in the state vector:*

$$\begin{aligned}\dot{x} &= -\alpha x, & \dot{y} &= \rho y - \kappa x, & x(0) &= x_0, \\ u &= \begin{pmatrix} x \\ y \end{pmatrix}, & \mathbf{A} &= \begin{pmatrix} -\alpha & 0 \\ -\kappa & \rho \end{pmatrix}, & f &= 0, & \Gamma &= \begin{pmatrix} 1 & 0 \end{pmatrix}, & d &= x_0.\end{aligned}\tag{4.10a}$$

Write $Q = \kappa/(\rho + \alpha)$, the coefficient q of (2.2) in this special case. The split is

$$\begin{aligned}E_s &= \text{span} \left\{ \begin{pmatrix} 1 \\ Q \end{pmatrix} \right\}, & E_u &= \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \\ \mathcal{P}_s \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x \\ Qx \end{pmatrix}, & \mathcal{P}_u \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ y - Qx \end{pmatrix}.\end{aligned}\tag{4.10b}$$

The general path separates the fundamental valuation from the growing valuation discrepancy:

$$\begin{aligned} x(t) &= x_0 e^{-\alpha t}, \\ y(t) &= Qx_0 e^{-\alpha t} + (y_0 - Qx_0) e^{\rho t}. \end{aligned} \quad (4.10c)$$

Admissibility selects $y_0 = Qx_0$, as in the first panel of Figure 2. The map Γ_s sends (a, Qa) to a , so it is bijective and $s_0 = (x_0, Qx_0)$. There is no forced component in this full system.

Eliminating income moves its effect into the forcing. Solve $x(t) = x_0 e^{-\alpha t}$ first and substitute it into the valuation equation. The reduced operator is now $A = \rho$, with $E_s = \{0\}$ and $\mathcal{P}_u = I$, but the forcing is nonzero:

$$\begin{aligned} \dot{y}(t) &= \rho y(t) + f(t), \quad f(t) = -\kappa x_0 e^{-\alpha t}, \\ (\mathcal{K}f)(t) &= h_u(t) = - \int_t^\infty e^{\rho(t-r)} f(r) dr = \int_t^\infty e^{-\rho(r-t)} \kappa x(r) dr = Qx(t), \\ r_f &= Qx_0, \quad r_f + E_s = \{Qx_0\}. \end{aligned} \quad (4.10d)$$

Here h_u is the forced unstable component used in (5.4). It is determined by future payouts, and differs from the freely chosen explosive discrepancy in (4.10c). The fundamental valuation that belonged to the full system's stable line is now supplied entirely by the Green response.

The reduced problem inherits no separate restriction on y . Its initial-data space is $D = \{0\}$, and $\Gamma_s : \{0\} \rightarrow \{0\}$ remains bijective. Thus a zero stable space need not imply nonexistence: there is simply no free stable adjustment to add to the unique forced valuation. If an additional condition prescribes $y(0) = y_0$, however, existence requires

$$y_0 = r_f = \int_0^\infty e^{-\rho t} \kappa x(t) dt = Qx_0. \quad (4.10e)$$

No stable adjustment can repair a failure of this present-value restriction. The elimination is legitimate because income evolves independently of valuation. With feedback, including $\beta > 0$ in (2.1), its future path must instead be solved jointly with the forward-looking variable. Appendix C.4 gives the derivation.

Existence asks whether any admissible adjustment can correct the required initial mismatch. Uniqueness asks whether two different admissible adjustments can correct the same mismatch. For a particular experiment, only its own $d - \Gamma r_f$ must belong to $\text{Ran } \Gamma_s$. Surjectivity of Γ_s is stronger: every inherited-state mismatch can then be accommodated. Example 1 permits every inherited income state when the valuation can adjust. Prescribing that valuation as well removes the adjustment margin and creates the compatibility restriction (4.10e). In a larger model, the existence test asks whether the available adjustments make the inherited economy and the announced future mutually consistent.

4.3 Stable and unstable subspaces

The stable-trace criterion in Section 4.2 requires a stable–unstable decomposition of equilibrium dynamics. We now construct that decomposition from household response factors, proceeding from a known fixed-price split to price feedback and then to the equilibrium spectral projection.

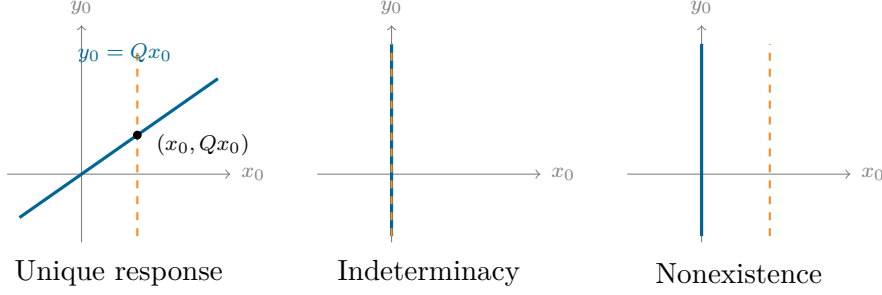


Figure 2: Equilibrium as saddle-path matching. Blue lines show admissible homogeneous initial combinations. Dashed orange lines impose the inherited state. The first panel represents Example 1: the stable line $y_0 = Qx_0$ pairs income capacity with its fundamental valuation, and the black intersection selects the unique response. The other panels depict a different system whose stable line is vertical. Coincident lines leave a free response. Parallel distinct lines give nonexistence. The last two panels share the same stable dynamics and differ only in the prescribed initial data.

4.3.1 Fixed-price household dynamics and the baseline split

Theorem 2 turns an infinite-horizon question into a boundary calculation for a stationary realization on a function space. The main computational obstacle is constructing its stable space when household values and distributions have many coordinates. The household factors used in fake news provide a way to apply the spectral projections without storing a full stable basis. We first identify the fixed-price household split. Section 4.3.2 closes the price-feedback loop, Section 4.3.3 constructs the equilibrium projection, and Section 5.2 applies it to exact continuation. Appendix A.2 gives the initial-data calculation. This use of household response objects is consistent with the recovery of state-space dynamics from sequence-space solutions in Auclert et al. (2021), Appendix C.7. The object constructed here is the stable–unstable projection of the complete equilibrium realization.

We work next on real Hilbert spaces E_v, E_g . The generator domains include the spatial boundary conditions, and conserved distribution directions are removed from E_g . Assume that the closed, densely defined household operators $\mathcal{L}_v$ and $\mathcal{L}_g$ generate strongly continuous semigroups with $\|e^{t\mathcal{L}_j}\| \leq M_j e^{-\alpha_j t}$, $j = v, g$, where $\alpha_j > 0$. Distribution disturbances then decay at fixed prices, and the discounted value semigroup attenuates distant price news. Assume the maps $\mathbf{B}_{gv} : E_v \rightarrow E_g$, $\mathbf{B}_{vp} : \mathbb{R}^{n_p} \rightarrow E_v$, $\mathbf{B}_{gp} : \mathbb{R}^{n_p} \rightarrow E_g$, and $P_g : E_g \rightarrow \mathbb{R}^{n_p}$ are bounded after the linearized spatial boundary equations have been incorporated. The number of aggregate prices n_p is finite. The household spaces need not be.

These are conditions on the chosen norms and domains, rather than boundedness of differential expressions on arbitrary density spaces. Appendix C.8 gives a weak-measure construction, proves a borrowing-floor mixing estimate, and states the additional flux regularity required for the coupling. On a fixed grid the semigroup conditions reduce to Hurwitz household matrices. Setting price feedback to zero gives an operator with domain $D(\mathbf{A}_0) = D(\mathcal{L}_v) \times D(\mathcal{L}_g)$:

$$\mathbf{A}_0 = \begin{pmatrix} -\mathcal{L}_v & 0 \\ \mathbf{B}_{gv} & \mathcal{L}_g \end{pmatrix}, \quad \Xi = \int_0^\infty e^{\mathcal{L}_g t} \mathbf{B}_{gv} e^{\mathcal{L}_v t} dt, \quad \mathcal{L}_g \Xi + \Xi \mathcal{L}_v = -\mathbf{B}_{gv}. \quad (4.11)$$

The Sylvester identity holds on $D(\mathcal{L}_v)$, and $\Xi D(\mathcal{L}_v) \subset D(\mathcal{L}_g)$. Here $\mathbf{A}_0$ contains both household equations and it is distinct from the distribution generator $\mathcal{L}_g$. The key idea of the third formula in (4.11) is to separate out a distribution profile g_u that moves together with v , such that $g_u = \Xi v$. The economic meaning is clearer in the equivalent expression:

$$\underbrace{\mathcal{L}_g \Xi v + \mathbf{B}_{gv} v}_{\dot{g}_u} = \underbrace{\Xi(-\mathcal{L}_v v)}_{\Xi \dot{v}}.$$

The left side combines the evolution of the existing distribution profile with the new redistribution caused by saving responses. The right side is the distribution change needed to remain associated with the evolving value profile. At fixed prices, stable directions have zero value deviation, while unstable value deviations carry the associated distribution profile Ξv . Thus

$$\begin{aligned} (v, g) &= (0, g - \Xi v) + (v, \Xi v), \\ \mathcal{P}_s^0(v, g) &= (0, g - \Xi v), \quad \mathcal{P}_u^0(v, g) = (v, \Xi v). \end{aligned} \tag{4.12}$$

These are complementary spectral projections, generally oblique.

The map Ξ records the distributional legacy associated with a value-function deviation at fixed aggregate prices. A change in the marginal value of wealth changes saving incentives. $\mathbf{B}_{gv}$ converts the policy response into a source in the distribution equation. To interpret the integral in (4.11), fix today's value profile v . At a hypothetical earlier date $-\tau$, the homogeneous value profile that evolves into v is $e^{\mathcal{L}_v \tau} v$. The induced distribution source is $\mathbf{B}_{gv} e^{\mathcal{L}_v \tau} v$, and $e^{\mathcal{L}_g \tau}$ propagates that source to today. Integrating over $\tau \geq 0$ gives Ξv , a signed distribution profile with zero total mass. This hypothetical history defines the invariant graph.

The subtraction $g - \Xi v$ in (4.12) removes the distribution component associated with the value deviation. Indeed, in the homogeneous fixed-price system, the transformed distribution coordinate $\eta := g - \Xi v$ satisfies

$$\dot{\eta} = \mathcal{L}_g \eta, \quad \dot{v} = -\mathcal{L}_v v, \quad g = \eta + \Xi v.$$

η captures the residual after subtracting the component Ξv coupled to the value dynamics. It therefore evolves under the baseline distribution dynamics $\mathcal{L}_g$, without a separate value-induced source $\mathbf{B}_{gv} v$. Thus $(0, g - \Xi v)$ describes redistribution that relaxes under the baseline household dynamics, while $(v, \Xi v)$ is the joint value and distribution movement along the unstable graph. Without forcing, the latter component must vanish for the path to be admissible. Both pieces contain a distribution component: simply assigning g to the stable space and v to the unstable space would ignore the saving response encoded in $\mathbf{B}_{gv}$. Appendix C.9 verifies the cancellation that separates these dynamics.

4.3.2 Fake news and equilibrium price feedback

Write $s \in \mathbb{C}$ for a complex frequency. A mode $e^{st} u_s$ turns a constant-coefficient dynamic equation into a shifted operator equation. The resolvent $(sI - \mathbf{A})^{-1}$, when it exists, maps a forcing amplitude at frequency s into the corresponding response amplitude. The household resolvents below perform the same calculation before price feedback is imposed. Appendix C.5 gives the Laplace-transform derivation and its relation to index criteria.

Let n_p be the number of price inputs. Replace the generic aggregation map $\mathcal{O}_g$ in (D.2) by the equilibrium-price map P_g , and write τ for the anticipation lag to distinguish it from the complex frequency s :

$$\begin{aligned}
F_p(\ell, \tau) &= P_g e^{\mathcal{L}_g \ell} \mathbf{B}_{gv} e^{\mathcal{L}_v \tau} \mathbf{B}_{vp}, \\
\widehat{F}_p(s) &= \int_0^\infty \int_0^\infty e^{-s\ell + s\tau} F_p(\ell, \tau) d\tau d\ell \\
&= -P_g (sI - \mathcal{L}_g)^{-1} \mathbf{B}_{gv} (sI + \mathcal{L}_v)^{-1} \mathbf{B}_{vp}, \\
M(s) &= I - P_g (sI - \mathcal{L}_g)^{-1} \mathbf{B}_{gp} - \widehat{F}_p(s).
\end{aligned} \tag{4.13}$$

If the household decay rates are $\alpha_g, \alpha_v > 0$, the integrals converge for $-\alpha_g < \text{Re } s < \alpha_v$, including the imaginary axis. The opposite exponential signs reflect forward distribution propagation and backward anticipation.

To interpret $M(s)$, consider a proposed price mode $\delta p(t) = e^{st} p$. The vector p specifies the relative price movements. The real part of s determines growth or decay, and its imaginary part determines oscillation. The three terms in $M(s)$ compare this proposal with the prices implied by the household response:

1. I retains the proposed price vector itself, so $I p = p$.
2. $P_g (sI - \mathcal{L}_g)^{-1} \mathbf{B}_{gp}$ gives the *direct saving channel*. The proposed prices change income and saving through $\mathbf{B}_{gp}$. The household distribution responds through $(sI - \mathcal{L}_g)^{-1}$. And P_g maps that redistribution back into equilibrium prices. This channel holds the value-function input fixed when differentiating saving.
3. $\widehat{F}_p(s)$ gives the *value and policy channel*. Anticipated prices change household values, values change consumption and saving, and the resulting distribution movement changes equilibrium prices. The transformed fake-news kernel collects this indirect response across anticipation and distribution-propagation lags.

Consequently,

$$M(s)p = p - \underbrace{\left[P_g (sI - \mathcal{L}_g)^{-1} \mathbf{B}_{gp} + \widehat{F}_p(s) \right]}_{\text{prices implied by the household response}} p.$$

Thus $M(s)p$ is a market consistency operator. It compares a proposed price movement with the price movement justified by households' response. Its inverse describes how prices must adjust to clear markets. In Aiyagari (1994), market clearing requires household asset supply to equal firms' capital: $\delta K^s = \delta K$. A proposed increase in capital changes interest rates and wages. Households respond to those prices by changing saving, which changes asset supply:

$$\text{capital} \rightarrow \text{interest rates and wages} \rightarrow \text{saving} \rightarrow \text{asset supply}.$$

The question is whether this loop returns the same capital movement that we initially proposed. Suppose that, for a particular time pattern, the shock directly generates one unit of additional asset supply. Each unit of proposed capital generates another 0.6 units of asset supply through

its effect on prices and household behavior. Then household asset supply is $\delta K^s = 1 + 0.6\delta K$. Market clearing requires

$$\delta K = 1 + 0.6\delta K \quad \text{or} \quad (1-0.6)\delta K = 1.$$

Here the scalar version of $M(s)$ is

$$m_K(s) = 1 - 0.6 = 0.4.$$

It measures the gap between proposed capital and the capital reproduced by household behavior, per unit of the proposal. With several prices, the matrix also accounts for feedback across prices. A small singular value signals a price direction in which this consistency calculation is sensitive to forcing and numerical error.

When the household resolvents exist, $M(s)p = 0$ for some $p \neq 0$ identifies a price pattern reproduced without external forcing, together with its associated household mode. Its growth or decay is determined by $\text{Re } s$. Under the semigroup and price-map regularity conditions below, Proposition 1 shows that nonsingularity of $M(i\omega)$ for every real ω excludes the entire imaginary-axis spectrum and permits a stable-unstable split. The small price solve in (4.15) then uses $M(s)$ to reconstruct the full equilibrium resolvent. Integrating its feedback correction in (4.16) constructs the spectral projections.

4.3.3 Constructing the equilibrium projection

In the last subsection, we see that the household responses used to accumulate economic effects into SSJ Jacobians also contain the information needed to construct the admissible equilibrium subspace. The subsequent Γ_s calculation matches that subspace to the particular distribution and other predetermined conditions the economy actually inherits. The household response factors and price-consistency matrix now supply the equilibrium resolvent. Its frequency integral constructs the stable projection.

The aggregate kernel must be retained together with the household maps that reconstruct values and distributions. For each frequency, compute the response maps from the n_p -dimensional price space below. The calligraphic $\mathcal{Y}$ denotes household response maps, while Y denotes aggregate output.

$$\begin{aligned} \mathcal{Y}_v(s) &= -(sI + \mathcal{L}_v)^{-1} \mathbf{B}_{vp}, \\ \mathcal{Y}_g(s) &= (sI - \mathcal{L}_g)^{-1} \{ \mathbf{B}_{gp} + \mathbf{B}_{gv} \mathcal{Y}_v(s) \}, \\ \mathcal{Y}(s) &= \begin{pmatrix} \mathcal{Y}_v(s) \\ \mathcal{Y}_g(s) \end{pmatrix}, \quad M(s) = I - P_g \mathcal{Y}_g(s). \end{aligned} \tag{4.14}$$

For an arbitrary input profile $x = (x_v, x_g)$, first solve the household equations at fixed prices, then impose price consistency:

$$\begin{aligned} v^0 &= (sI + \mathcal{L}_v)^{-1} x_v, & g^0 &= (sI - \mathcal{L}_g)^{-1} (x_g + \mathbf{B}_{gv} v^0), \\ M(s)p &= P_g g^0, & (sI - \mathbf{A})^{-1} x &= \begin{pmatrix} v^0 \\ g^0 \end{pmatrix} + \mathcal{Y}(s)p. \end{aligned} \tag{4.15}$$

The superscript 0 denotes this auxiliary fixed-price frequency response, not an initial condition. Each evaluation uses household resolvent actions and a small price solve.

On $E = E_v \times E_g$, let $U_p = (-\mathbf{B}_{vp}, \mathbf{B}_{gp})^\top$ and $V_p = (0 \ P_g)$. Require $U_p : \mathbb{R}^{n_p} \rightarrow D(\mathbf{A}_0)$ to be bounded for the graph norm, and require $V_p \mathbf{A}_0$, initially defined on $D(\mathbf{A}_0)$, to extend boundedly to $E \rightarrow \mathbb{R}^{n_p}$. A unit price input must therefore have a well-defined household time derivative, and the instantaneous change in the observed aggregate must be controlled by the household state norm. These sufficient regularity conditions are automatic for matrices. In a differential realization they are additional hypotheses. Neither stationarity nor the borrowing-floor mixing argument establishes them.

Proposition 1 (Stable projections from household responses). *Under the preceding household semigroup, bounded-coupling, and price-map regularity conditions, let $\mathbf{A} = \mathbf{A}_0 + U_p V_p$ on $D(\mathbf{A}_0)$. The following are equivalent:*

1. $M(i\omega)$ is nonsingular for every $\omega \in \mathbb{R}$.
2. The imaginary axis is contained in the resolvent set of $\mathbf{A}$, so it contains no point, continuous, or residual spectrum.
3. $\mathbf{A}$ admits an exponential dichotomy.

When these conditions hold, define

$$\begin{aligned} \mathcal{R}_0(s) &= (sI - \mathbf{A}_0)^{-1}, \\ \Delta \mathcal{R}(s) &= \mathcal{Y}(s) M(s)^{-1} (0 \ P_g) \mathcal{R}_0(s), \\ \mathcal{P}_s &= \mathcal{P}_s^0 + \frac{1}{\pi} \int_0^\infty \operatorname{Re}\{\Delta \mathcal{R}(i\omega)\} d\omega, \quad \mathcal{P}_u = I - \mathcal{P}_s. \end{aligned} \tag{4.16}$$

The integral converges in bounded-operator norm on E , and $\mathcal{P}_s$ projects onto the stable invariant space along the unstable invariant space. No smallness restriction on price feedback is required. This supplies the dynamic part of Assumption 2. Closed range of the initial-data map remains a separate condition.

Appendix C.9 proves the proposition and gives bounds for quadrature and for certifying frequency separation.

The baseline projection $\mathcal{P}_s^0$ describes household dynamics with aggregate prices held fixed. It already includes the effect of values on saving and redistribution through $\mathbf{B}_{gv}$ and Ξ . General equilibrium closes the return channel: redistribution changes prices, which feed back into values and saving. This changes which combinations of values and distributions converge. At fixed prices, decaying departures have $v = 0$. In Aiyagari, an inherited increase in aggregate capital lowers interest rates and raises wages. Households' current valuations must reflect the subsequent equilibrium price path. A convergent equilibrium transition therefore generally requires an initial value adjustment. The correction $\mathcal{P}_s - \mathcal{P}_s^0$ incorporates this price feedback into the admissible directions and vanishes when $P_g = 0$.

Computationally, the decomposition reuses the known household split and the response factors underlying fake news. For an input profile x , first compute the fixed-price response

$\mathcal{R}_0(s)x = (v^0, g^0)$. Its distribution component implies the price movement $P_g g^0$. Solving $M(s)p = P_g g^0$ makes the price response consistent with household feedback, and $\mathcal{Y}(s)p = \Delta \mathcal{R}(s)x$ reconstructs the additional value and distribution responses. Integrating this correction as in (4.16) gives the action of the equilibrium stable projection. $\mathcal{P}_u x = x - \mathcal{P}_s x$ gives the complementary action. Thus household solves and small price solves construct the split without storing a full stable basis. These actions enter the initial-data calculation in (A.3) and the continuation condition in (5.10). The baseline term retains household directions that are invisible to price experiments. This is why an aggregate fake-news table alone does not identify the complete stable space.

5 IRFs, exact continuation and the aggregate SSJ representation

Section 4 establishes the determinacy criterion and constructs the stable and unstable subspaces from household responses. We now connect this representation to the aggregate sequence-space closure. Regular elimination preserves the existence and uniqueness restrictions. The same projections then impose inherited initial data and the admissible continuation after a finite matching date, retaining future price feedback within the SSJ calculation.

5.1 Aggregate sequence-space closure

The stable-space calculation is an internal realization of the sequence-space equilibrium. Household value and distribution paths can be eliminated to retain only aggregate paths as equilibrium unknowns. Analytical fake news computes the derivatives of that elimination, as developed in Appendix D.1. Write w_{int} for internal *deviation* paths and retain δp as aggregate unknowns. Starting from (4.1), retain the price path as the aggregate unknown and solve the household equations conditional on it. For $g(0) = 0$, the first two rows give

$$\underbrace{\begin{pmatrix} v \\ g \end{pmatrix}}_{w_{\text{int}}} = \underbrace{\begin{pmatrix} -\partial_t - \mathcal{L}_v & -\mathbf{B}_{vg} \\ -\mathbf{B}_{gv} & \partial_t - \mathcal{L}_g \end{pmatrix}^{-1}}_{\mathcal{A}^{-1}} \left[\underbrace{\begin{pmatrix} b_v \\ b_g \end{pmatrix}}_{b_w} + \underbrace{\begin{pmatrix} \mathbf{B}_{vp} \\ \mathbf{B}_{gp} \end{pmatrix}}_{-\mathcal{B}} \delta p \right].$$

The displayed abbreviations group the original household blocks: $\mathcal{B}$ is their negative price column. We assume that $\mathcal{A}$ has a bounded inverse on the chosen path spaces, with homogeneous initial-distribution data, spatial boundary conditions, and admissibility at infinity included in its domain. The inverse therefore solves for entire value and distribution paths.

Writing $\mathcal{C} = (\mathcal{C}_v \ \mathcal{C}_g)$ and $\mathcal{E} = \mathcal{C}_p$, the market equation $\mathcal{C}_v v + \mathcal{C}_g g + \mathcal{C}_p \delta p = b_p$ completes the same system in compact form:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{E} \end{pmatrix} \begin{pmatrix} w_{\text{int}} \\ \delta p \end{pmatrix} = \begin{pmatrix} b_w \\ b_p \end{pmatrix}. \quad (5.1)$$

Substituting the household solution into market clearing gives the aggregate SSJ closure:

$$\mathcal{J}_{\text{GE}} = \mathcal{C}_p + \begin{pmatrix} \mathcal{C}_v & \mathcal{C}_g \end{pmatrix} \mathcal{A}^{-1} \begin{pmatrix} \mathbf{B}_{vp} \\ \mathbf{B}_{gp} \end{pmatrix} = \mathcal{E} - \mathcal{C} \mathcal{A}^{-1} \mathcal{B}, \quad (5.2)$$

$$\mathcal{J}_{\text{GE}} \delta p = b_p - \mathcal{C} \mathcal{A}^{-1} b_w.$$

Read the household contribution from right to left: proposed prices change payoffs and distributional flows. The household solve determines the resulting paths. The market maps measure their effect on clearing. Its plus sign follows from moving the negative price column in (4.1) to the right. A nonzero inherited distribution deviation adds its household response and subtracts the associated market-residual contribution on the right of (5.2). The price-path operator is unchanged.

Proposition 2 (Equivalence of full and reduced closure). *If $\mathcal{A}$ is boundedly invertible, the full operator in (5.1) is equivalent by bounded invertible row and column operations to $\text{diag}(\mathcal{A}, \mathcal{J}_{\text{GE}})$. Therefore kernel, cokernel, and Fredholm index of the equilibrium problem equal those of the aggregate SSJ closure, up to canonical isomorphisms.*

Changing the aggregate unknowns preserves determinacy when elimination and reconstruction are valid. Appendix C.6 proves Proposition 2 and states the additional assumptions required for local nonlinear uniqueness.

5.2 Exact continuation and aggregate impulse response

Section 4 characterizes equilibrium by intersecting two restrictions at date zero. The inherited distribution fixes predetermined states, and admissibility fixes the combination of jump variables consistent with a nonexplosive future. The same argument can be applied at a candidate response up to T . Its date- T state must be one that can be continued as an equilibrium after T . We therefore need the set of admissible equilibrium states at that date, rather than an arbitrary terminal value for each variable.

For stationary linearized coefficients and no remaining forcing, that set is the stable invariant subspace of the complete equilibrium generator. With future shocks, it is a parallel affine set shifted by the particular response to those shocks. Stable directions summarize permissible adjustment from an inherited state. Unstable directions pin down the jump variables needed to avoid an explosive continuation. This is the same economic saddle-path selection used in the determinacy test. The initial-data map asks whether the selection works at date zero. The tail map applies the selection at date T .

Theorem 2 selects the initial equilibrium response by matching stable adjustments to inherited data. The same equilibrium projections determine the admissible continuation at a finite matching date T . Suppose the reduced equilibrium dynamics beyond T are

$$\dot{u} = \mathbf{A}_\infty u + f(t), \quad t \geq T, \quad (5.3)$$

with the dichotomy of Assumption 2. The stable trace can adjust to the distribution and other states inherited at T , whereas future forcing determines the unstable trace:

$$h_u(T) = - \int_T^\infty e^{\mathbf{A}_\infty(T-r)} \mathcal{P}_u f(r) dr. \quad (5.4)$$

The exact continuation condition is

$$\boxed{\mathcal{P}_u u(T) = h_u(T)}. \quad (5.5)$$

It is imposed in $E_u = \text{Ran } \mathcal{P}_u$, without counting redundant coordinate rows as additional restrictions. With no remaining forcing, it requires $u(T) \in E_s$. With future shocks, it selects a parallel affine set. This is the date- T version of the saddle-path selection used at date zero: the same projection selects admissible initial adjustments and terminal continuation, including all future forcing.

Theorem 3 (Tail-exact equivalence). *Assume the stationary tail satisfies Assumption 2 and $f \in L^2([T, \infty); E)$. Restriction to $[0, T]$ gives a bijection between admissible half-line solutions and prefix solutions satisfying the original initial conditions and (5.5). Each compatible terminal trace has a unique admissible continuation. Consequently, exact prefix solution and exact continuation introduce no horizon truncation error, independently of the matching date within the stationary region. If the complete half-line problem has a unique admissible response, every such formulation recovers that response.*

Appendix D.4 gives the continuation formula and proof.

Consider a finite computation on $[0, T]$. Setting forward-looking variables to zero at T generally replaces the correct admissible-trace relation by a different boundary condition. Even after forcing has ended, the stable equilibrium path typically remains nonzero. Its forward-looking and predetermined components must satisfy a stable-subspace relation, not componentwise disappearance.

Three dates must be distinguished. The shock announcement date sets the economic information boundary. The last date displayed in an impulse-response figure is a reporting choice. The matching date T is an interface between an explicitly represented prefix and an analytically continued tail. Only the first is an intrinsic part of the model. Choosing the latter two should not change the equilibrium path.

Start from the aggregate equilibrium equation $\mathcal{J}_{GE} \delta p = b$, with household paths eliminated as in (5.2). The right side includes shocks and inherited-state effects. Split the price path into the explicit interval $E = [0, T]$ and the infinite future. Let $\mathcal{J}_{EE}, \mathcal{J}_{E\infty}, \mathcal{J}_{\infty E}$, and $\mathcal{J}_{\infty\infty}$ denote the corresponding blocks of this same half-line equilibrium operator:

$$\begin{pmatrix} \mathcal{J}_{EE} & \mathcal{J}_{E\infty} \\ \mathcal{J}_{\infty E} & \mathcal{J}_{\infty\infty} \end{pmatrix} \begin{pmatrix} \delta p_E \\ \delta p_\infty \end{pmatrix} = \begin{pmatrix} b_E \\ b_\infty \end{pmatrix}. \quad (5.6)$$

If the future block is boundedly invertible on the specified spaces, then $\delta p_\infty = \mathcal{J}_{\infty\infty}^{-1}(b_\infty - \mathcal{J}_{\infty E} \delta p_E)$. Substitution gives

$$\begin{aligned} \mathcal{S}_T &= \mathcal{J}_{EE} - \mathcal{J}_{E\infty} \mathcal{J}_{\infty\infty}^{-1} \mathcal{J}_{\infty E}, \\ \mathcal{S}_T \delta p_E &= b_E - \mathcal{J}_{E\infty} \mathcal{J}_{\infty\infty}^{-1} b_\infty. \end{aligned} \quad (5.7)$$

The unknowns are still aggregate price paths on the explicit interval. The correction retains a feedback that a zero tail would remove: earlier prices change the distribution at T . That distribution changes future equilibrium prices. Anticipation of those prices changes decisions

before T . The right-side correction also retains announced shocks after T . Even if those shocks have ended, the Jacobian correction generally remains because endogenous adjustment continues. Moreover, $\mathcal{J}_{EE}$ is a restriction of the correctly defined half-line operator. A household Jacobian computed with an arbitrary terminal policy need not equal that block.

Equation (5.7) is an exact sequence-space identity, but the future inverse still needs to be evaluated. The dynamic realization supplies that evaluation through an admissible terminal relation. When working with the differential equations, all initial, spatial, and interface trace conditions must accompany the block elimination. If the future block is singular, one must retain its free directions and compatibility conditions. The projection condition (5.5) remains available without that inverse.

The continuation formula (D.7) makes this elimination constructive. Evaluating it at $t = T$ gives $u(T) = \mathcal{P}_s u(T) + h_u(T)$, hence $\mathcal{P}_u u(T) = h_u(T)$. Future prices therefore need not be stored at all future dates: their effect on earlier household decisions enters through the terminal values compatible with this condition. These values must be solved together with the prefix response, since they change the distribution that reaches T .

For an explicit construction, take the stationary household block (3.8), with $\mathbf{B}_{vg} = 0$, and reduced state $u = (v, g)$. For each trial price path δp_E , first solve the household equations on $E = [0, T]$ with inherited distribution g_0 , all given interior forcing (b_v, b_g) , and the reference terminal condition $v^E(T) = 0$. Denote the resulting paths by (v^E, g^E) . The superscript E identifies this zero-terminal reference calculation. It does not impose zero terminal deviations on the equilibrium. Appendix D.7 gives its explicit integrals. If $v_T = v(T)$ is the continuation value still to be determined, variation of constants gives

$$\begin{aligned} v(t) &= v^E(t) + e^{\mathcal{L}_v(T-t)} v_T, \\ g(t) &= g^E(t) + \int_0^t e^{\mathcal{L}_g(t-r)} \mathbf{B}_{gv} e^{\mathcal{L}_v(T-r)} v_T dr. \end{aligned} \quad (5.8)$$

The added term in v transmits future incentives backward. Its effect on saving then accumulates forward in g . At the matching date this yields

$$\Xi_T := \int_0^T e^{\mathcal{L}_g(T-r)} \mathbf{B}_{gv} e^{\mathcal{L}_v(T-r)} dr, \quad g(T) = g^E(T) + \Xi_T v_T. \quad (5.9)$$

Thus Ξ_T maps a terminal value profile into the change in the terminal distribution caused by earlier saving responses. Under the household stability assumptions defining Ξ , it satisfies $\Xi_T = \Xi - e^{\mathcal{L}_g T} \Xi e^{\mathcal{L}_v T}$. This is the finite-horizon version of the same policy-to-distribution propagation used by the fake-news construction.

Substitute this terminal distribution into the projection condition:

$$\mathcal{P}_u \begin{pmatrix} v_T \\ \Xi_T v_T \end{pmatrix} = h_u(T) - \mathcal{P}_u \begin{pmatrix} 0 \\ g^E(T) \end{pmatrix}. \quad (5.10)$$

For the given trial prices, $g^E(T)$ is known. When the interface operator on the left is boundedly invertible, solving for v_T closes the household response while retaining all future forcing through $h_u(T)$. Projection actions suffice. A stable basis or a stored policy map is unnecessary. The equation retains all initial and spatial boundary conditions.

When the stable trace onto the distribution is bijective, the stable space has the policy-rule form $E_s = \{(Qg, g)\}$. Writing $h_u(T) = (h_v(T), h_g(T))$, the same terminal condition becomes

$$\begin{aligned} v_T &= Qg(T) + h_v(T) - Qh_g(T), \\ (I - Q\Xi_T)v_T &= Qg^E(T) + h_v(T) - Qh_g(T). \end{aligned}$$

The first equality selects values consistent with the inherited distribution and future shocks. The second accounts for the effect of those values on the distribution itself. Eliminating v_T requires this interface solve to be boundedly invertible.

If a graph representation or the interface inverse is unavailable, retain v_T as an auxiliary unknown and impose $\mathcal{P}_u(v_T, g(T)) = h_u(T)$ together with the prefix market residuals. For models with additional aggregate states, include their traces in this condition. This augmented formulation preserves free tail directions and compatibility restrictions. One must not replace a singular interface solve by an arbitrary inverse or impose a zero terminal value to remove it. Using $g(T) = g^E(T) + \Xi_T v_T$ gives the interface equation (5.10). Its left side combines the household terminal-response action with the frequency construction of $\mathcal{P}_u$. It can be included in an augmented iterative SSJ solve, together with its adjoint, without forming Q . The residual belongs to E_u . Redundant full-coordinate rows are not additional equilibrium restrictions. If this interface elimination is regular, it defines the same aggregate Jacobian action as the graph solve. Otherwise the augmented equations retain its null directions and compatibility conditions.

Once the terminal value has been obtained, substitute the corrected paths (5.8) into the prefix market equations. When the household and future eliminations are regular, the resulting residual is

$$\begin{aligned} R_T(\delta p_E) &:= \mathcal{C}_v v + \mathcal{C}_g g + \mathcal{C}_p \delta p_E - b_p \\ &= \mathcal{S}_T \delta p_E - (b_E - \mathcal{J}_{E\infty} \mathcal{J}_{\infty\infty}^{-1} b_\infty), \\ \mathcal{S}_T \delta p_E &= R_T(\delta p_E) - R_T(0). \end{aligned} \tag{5.11}$$

All terms in the first line are restricted to the prefix. The household paths use the same inherited data and forcing in both residual evaluations. Equivalently, a Jacobian action sets those exogenous inputs to zero, applies a trial price path, and repeats the terminal-consistency solve. To obtain the corrected right side, instead set trial prices to zero, retain all given initial data and shocks, and take $-R_T(0)$. Solve $\mathcal{S}_T \delta p_E = -R_T(0)$ for the equilibrium prefix, either by assembling these actions or with an iterative linear solver. This constructs both parts of (5.7) without explicitly inverting an infinite future-price block.

Once δp_E has been solved from (5.11), recover the remaining aggregate IRFs by evaluating the household block at this equilibrium price path. Recompute, or reuse, v^E, g^E with the actual inherited distribution and shock inputs. Obtain v_T from (5.10), or take it from the augmented solve when retained as an unknown. Equation (5.8) then gives the equilibrium value and distribution paths $v(t), g(t)$. For an observable Y with the aggregation maps of (3.8), its impulse response is

$$\delta Y(t) = \mathcal{O}_g g(t) + \mathcal{O}_v v(t) + \mathcal{O}_p \delta p_E(t), \quad 0 \leq t \leq T. \tag{5.12}$$

The first term aggregates redistribution across household states. The second aggregates policy changes induced by values. The third records direct price effects. Asset supply uses the asset moment of g , whereas consumption also reflects changed household policies. Any direct exogenous contribution to the observable must be added as well. Equation (6.19) displays the equivalent formula separating the reference aggregate from the terminal-value correction. These are evaluations of the solved equilibrium and require no additional market-clearing solve. The following proposition formalizes the equivalence to (5.7).

Proposition 3 (Sequence-space elimination with an exact tail). *Consider the same complete linearized equilibrium and admissible path spaces as in Theorem 3. Suppose that eliminating internal household paths and eliminating the future are both valid bounded block eliminations, including the initial and interface trace conditions. Whenever either order of elimination is used, the resulting equation for the same retained aggregate prefix paths is (5.7), with the same corrected forcing and the same reconstructed equilibrium. The kernel and cokernel of the complete problem are preserved by these eliminations. Under the global stationary dichotomy of Theorem 2, they are consequently isomorphic to those of Γ_s .*

Appendix D.7.1 derives the terminal correction and proves the proposition. These are statements about admissible path operators. In a numerical time-grid implementation, integration and linear solves introduce their own errors. Exact treatment of the tail removes the artificial horizon boundary without eliminating those errors. Section 6.3 supplies the Aiyagari specialization and Algorithm 2.

6 Computation and numerical illustrations

The continuous-space theory specifies what must be continued. The DEC scheme (Du, 2026) produces a finite-state realization with the same population accounting and backward–forward pairing. At mesh size h , form the closed-loop generator $A_{GE,h}$ on the mass-zero distribution space, not the unrestricted KFE matrix. Its stable–unstable projections are $\mathcal{P}_{s,h}, \mathcal{P}_{u,h}$. For the household feedback realization, Algorithm 1 computes their actions using the same discrete operators. (6.11) reduces its price step to a scalar in the Aiyagari example. A Schur calculation remains a useful benchmark on moderate grids. If the finite-state system is hyperbolic, the exact terminal restriction is

$$\mathcal{P}_{u,h}u_h(T) = - \int_T^\infty e^{A_{GE,h}(T-r)} \mathcal{P}_{u,h}f_h(r)dr. \quad (6.1)$$

Nothing in this expression discretizes or truncates time. The grid is in individual states. The terminal relation retains the entire future of the semi-discrete economic model. Numerical matrix-function evaluations introduce a tolerance. Time-horizon truncation is absent from the defining relation.

Corollary 3.1 (Tail matching for the DEC economy). *Fix a spatial mesh and consistently differentiate its household and market-clearing equations. Suppose the resulting reduced equilibrium generator has a stable–unstable dichotomy. Then solving its prefix with (6.1) and reconstructing*

the tail by (D.7) gives exactly the same solution set as the infinite-horizon, continuous-time economy on that mesh. In exact arithmetic, changing T does not change the solution. If the stable initial-data map is invertible, this solution is unique for every admissible forcing and initial datum.

Proof. Apply Theorems 2 and 3 to the finite-dimensional generator. The conservative derivative identities guarantee that its household distribution state and forcing remain in the mass-zero domain used in constructing the reduced system. The dichotomy and boundary rank, which are separate assumptions, supply admissibility and uniqueness. $\square$

The corollary is exact for the chosen spatial economy. It does not assert that its stable subspace equals a projection of the continuous-space stable subspace. Approximating the PDE still requires spatial consistency, differentiated consistency, and appropriate stability or spectral convergence. The useful inheritance is structural: the population constraint, dual aggregation, equilibrium residual, and continuation calculation all refer to the same discrete equations. One can therefore separate spatial approximation from terminal-horizon error instead of mixing them.

6.1 Aiyagari (1994): household responses and market clearing

The one-asset Aiyagari economy makes the entire procedure concrete. Households save in productive capital, face idiosyncratic labor-income risk, and cannot borrow below $\underline{a}$. A competitive firm rents capital and hires labor. We study the first-order response to an aggregate productivity path $\delta Z(t)$, announced at date zero, starting from a given distribution. Household fake-news factors construct the equilibrium spectral projection, which supplies both the determinacy test and the continuation condition for an aggregate SSJ solve. A full-equilibrium Schur decomposition is used afterwards as an independent numerical benchmark. Appendix C.10 gives the computational derivations, accuracy checks, and reproduction instructions.

Let z_i denote idiosyncratic productivity and let q_{ij} be the entries of its Markov generator, including diagonal entries with zero row sums. Period utility is $U(c)$, with $U' > 0$ and $U'' < 0$. The level HJB is

$$\rho V_i - \partial_t V_i = \max_c \left\{ U(c) + (wz_i + ra - c)\partial_a V_i + \sum_j q_{ij} V_j \right\}. \quad (6.2)$$

Given trial steady-state prices, solve for $\bar{V}_i$, consumption $\bar{c}_i$, and saving $\bar{s}_i = \bar{w}z_i + \bar{r}a - \bar{c}_i$. Their generator $\bar{L}$ determines the stationary distribution through $\bar{L}^* \bar{\mu} = 0$, with total mass one.

Let the capital share be $\theta \in (0, 1)$, depreciation be δ_{dep} , and aggregate labor be fixed at $\bar{N}$. In levels,

$$\begin{aligned} Y &= ZK^\theta \bar{N}^{1-\theta}, \\ r &= \theta ZK^{\theta-1} \bar{N}^{1-\theta} - \delta_{\text{dep}}, \quad w = (1 - \theta)ZK^\theta \bar{N}^{-\theta}. \end{aligned} \quad (6.3)$$

Writing depreciation with a subscript distinguishes that parameter from the perturbation operator δ . Stationary capital is $\bar{K} = \sum_i \int a \bar{\mu}_i(da)$. A finite asset moment is part of the admissible distribution space.

For the numerical illustration, set $U(c) = -1/c$, $\rho = \delta_{\text{dep}} = 0.05$, $\theta = 1/3$, and $\bar{Z} = 0.1$. Income takes values $z_1 = 1, z_2 = 2$, with switching intensities $q_{12} = q_{21} = 1/3$. The stationary income shares are one half each, so $\bar{N} = 3/2$.

At first order, the envelope condition removes the direct effect of δc from the HJB: its utility and saving effects offset at an interior optimum. The derivative of the first-order condition then recovers the consumption response. On a differentiable policy region,

$$\begin{aligned} (\rho - \bar{L} - \partial_t)v_i &= \partial_a \bar{V}_i(a \delta r + z_i \delta w), \\ \delta c_i &= \frac{\partial_a v_i}{U''(\bar{c}_i)}, & \delta s_i &= a \delta r + z_i \delta w - \delta c_i, \\ \dot{g}_i &= (\bar{L}^* g)_i - \partial_a(\bar{\mu}_i \delta s_i). \end{aligned} \tag{6.4}$$

The input $\delta p = (\delta r, \delta w)^\top$ enters the value equation through known payoff exposures $a \partial_a \bar{V}_i$ and $z_i \partial_a \bar{V}_i$. A higher expected wage or interest rate can therefore change consumption and saving before that price change occurs. The KFE translates the saving response into a redistribution of households. The response g has positive and negative components and integrates to zero. The state-constraint and flux conditions are differentiated consistently with these interior equations. At a borrowing boundary or switching point, we use the appropriate one-sided/weak response. The displayed division by U'' describes the smooth policy region.

To make the link with analytical fake news explicit, let $\mathbf{B}_{vp,r}$ and $\mathbf{B}_{vp,w}$ be the two payoff-exposure columns and let $\mathcal{L}_v = \bar{L} - \rho I$. The consumption response to a price change $j \in \{r, w\}$ occurring τ units ahead uses the policy derivatives $\mathbf{B}_{cv}, \mathbf{B}_{cp}$ defined in Section 3:

$$\begin{aligned} \phi_{c,j}(\tau) &= \mathbf{B}_{cv} e^{\mathcal{L}_v \tau} \mathbf{B}_{vp,j}, \\ \delta c(t) &= \mathbf{B}_{cp} \delta p(t) + \sum_{j \in \{r, w\}} \int_0^\infty \phi_{c,j}(\tau) \delta p_j(t + \tau) d\tau. \end{aligned} \tag{6.5}$$

On a smooth interior policy region, $\mathbf{B}_{cv} v = \partial_a v / U''(\bar{c})$. The calibrated calculation keeps the upwind active set fixed when differentiating. At zero-drift nodes the corresponding rule is $\delta c = a \delta r + z_i \delta w$, so the direct term $\mathbf{B}_{cp} \delta p$ is retained. This is a contemporaneous constrained policy response, distinct from anticipated news.

For anticipated saving news the induced distributional source is $\mathcal{D}_{\tau,j} = \partial_a(\bar{\mu} \phi_{c,j}(\tau))$, because the induced saving change is $-\delta c$. The direct source is $-\partial_a[\bar{\mu}(a \delta r + z_i \delta w - \mathbf{B}_{cp} \delta p)]$. These are the $\mathbf{B}_{gv}$ and $\mathbf{B}_{gp}$ channels in (3.8), with their discrete counterparts obtained from the same saving flux. Independent perturbations of values and prices check both the HJB and KFE derivatives, including the active boundary rule.

6.2 Stable projection and determinacy test

The preceding closure is expressed through continuous-state household operators. We construct its spatial realization using the discrete exterior calculus (DEC) scheme of Du (2026). DEC represents household values at primal nodes, saving fluxes on edges, and probability masses in dual cells. A shared incidence structure governs backward evaluation and forward transport, while dual-cell volumes provide the density-mass conversion on the nonuniform asset grid. In mass

coordinates, the forward KFE generator is the transpose of the undiscounted backward household generator. This construction preserves population accounting and the value–distribution pairing used to derive the household responses.

The subscript h denotes dependence on the spatial mesh. Thus v_h contains nodal value deviations, m_h contains cell probabilities, and $\bar{L}_h$ discretizes the backward generator $\bar{L}$. The matrix $\mathcal{L}_{g,0,h}$ realizes $\mathcal{L}_{g,0}$ in the reduced distribution coordinates defined below. h only concerns the discretization of individual states. Time t remains continuous at this stage.

The geometric reason for this choice is that distributional perturbations must remain in the tangent space of the unit-mass constraint, $\mathbf{1}^\top \delta m_h = 0$. Consistently differentiating the same conservative transport, including its boundary rules, keeps the policy and price sources in this space. Household fake news, aggregate capital, and the stable-trace map then refer to the same finite-grid equilibrium closure. The choice of DEC therefore preserves the conservation and duality structure needed by our geometric calculation. The spectral separation and boundary-rank conditions must still be checked for the resulting equilibrium generator.

Let $\bar{L}_h$ be the steady-state backward generator on n individual-state nodes. Use a matrix Q_m with orthonormal columns spanning the admissible distribution subspace and write $\delta m_h = Q_m \xi$, so $Q_m^\top Q_m = I$ and $\xi = Q_m^\top \delta m_h$. This basis encodes population-accounting constraints. Q_s spans stable equilibrium directions. Imposing total mass alone gives $q = n - 1$ independent coordinates. Here we also hold both income-state shares at their stationary values, consistently with fixed aggregate labor: perturbations sum to zero within each income state. This invariant subspace has $q = n - 2 = 2(N_a - 1)$ coordinates. Allowing income-share perturbations would require including their aggregate labor response in firm pricing. Denote by $\mathbf{B}_{gv,h}v$ the policy-induced cell-mass source and by $\mathbf{B}_{gp,h}\delta p$ the direct price-induced source, constructed by differentiating the same generator family. Define

$$\begin{aligned} \mathcal{L}_{v,h} &= \bar{L}_h - \rho I, & \mathcal{L}_{g,0,h} &= Q_m^\top \bar{L}_h^\top Q_m, \\ \mathbf{B}_{gv,0,h} &= Q_m^\top \mathbf{B}_{gv,h}, & \mathbf{B}_{gp,0,h} &= Q_m^\top \mathbf{B}_{gp,h}, \\ k_h^\top &= a_h^\top Q_m, & P_K &= (r_K, w_K)^\top, \quad P_Z = (r_Z, w_Z)^\top. \end{aligned} \tag{6.6}$$

Here $a_h \in \mathbb{R}^n$ lists the asset holding at each household-state node, with the asset grid repeated across the two income states. The superscript $\top$ denotes transpose: the row $a_h^\top$ aggregates cell probabilities into household asset supply, $K^s = a_h^\top m_h$. Because m_h contains cell probabilities, the cell-volume weights are already incorporated. Market clearing identifies asset supply with productive capital, so $\delta m_h = Q_m \xi$ gives

$$\delta K = a_h^\top \delta m_h = a_h^\top Q_m \xi = k_h^\top \xi.$$

Thus $k_h^\top \in \mathbb{R}^{1 \times q}$ is the same asset-aggregation map expressed in the reduced distribution coordinates. Its j th entry is the change in aggregate capital generated by one unit of the redistribution pattern in column j of Q_m . The row $a_h^\top$ acts on the full cell-mass vector, whereas $k_h^\top$ acts on its independent coordinates ξ .

Equation (6.7) is the matrix representation of (4.5). The two columns of $\mathbf{B}_{vp,h}$ are the discrete HJB payoff derivatives. Their formula and the policy derivative must come from the

same discretized optimality conditions. The implied price response is $\delta p = P_K k_h^\top \xi + P_Z \delta Z$. Substitution gives the discrete forward-backward representation

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} v_h \\ \xi \end{pmatrix} &= \underbrace{\begin{pmatrix} -\mathcal{L}_{v,h} & -\mathbf{B}_{vp,h} P_K k_h^\top \\ \mathbf{B}_{gv,0,h} & \mathcal{L}_{g,0,h} + \mathbf{B}_{gp,0,h} P_K k_h^\top \end{pmatrix}}_{\mathbf{A}_{GE,h}} \begin{pmatrix} v_h \\ \xi \end{pmatrix} + \underbrace{\begin{pmatrix} -\mathbf{B}_{vp,h} P_Z \\ \mathbf{B}_{gp,0,h} P_Z \end{pmatrix}}_{b_{Z,h}} \delta Z, \\ \Gamma_h \begin{pmatrix} v_h(0) \\ \xi(0) \end{pmatrix} &= \xi_0, \quad \Gamma_h = \begin{pmatrix} 0 & I \end{pmatrix}, \quad \xi_0 = Q_m^\top \delta m_h(0). \end{aligned} \quad (6.7)$$

The signs in the upper row follow from $-\dot{v}_h = \mathcal{L}_{v,h} v_h + \mathbf{B}_{vp,h} \delta p$. The lower row is the conservative KFE. This matrix and (D.18) represent the same household–firm closure whenever household elimination is valid.

For $N_a = 100$, there are 200 value coordinates, 198 distribution coordinates, and a 398-dimensional equilibrium realization. The implementation checks constant preservation, source-mass conservation, and invariance of the restricted distribution space. The full matrix is available for residual checks and moderate-grid time propagation, while the split below is constructed from household feedback.

The general price calculation in Section 4.3.2 simplifies further here. Although households respond to two prices, their endogenous changes both come from one scalar, aggregate capital: $P_g = P_K k_h^\top$. Consequently,

$$\mathbf{A}_{GE,h} = \begin{pmatrix} -\mathcal{L}_{v,h} & 0 \\ \mathbf{B}_{gv,0,h} & \mathcal{L}_{g,0,h} \end{pmatrix} + \begin{pmatrix} -\mathbf{B}_{vp,h} P_K \\ \mathbf{B}_{gp,0,h} P_K \end{pmatrix} \begin{pmatrix} 0 & k_h^\top \end{pmatrix}. \quad (6.8)$$

The price-feedback perturbation has rank at most one, although its effect on the stable space need not have rank one. The relevant fake-news kernel measures the asset-supply response to the price changes induced by a unit of trial capital:

$$F_K(\ell, \tau) = k_h^\top e^{\mathcal{L}_{g,0,h} \ell} \mathbf{B}_{gv,0,h} e^{\mathcal{L}_{v,h} \tau} \mathbf{B}_{vp,h} P_K. \quad (6.9)$$

Use the signed transform in (4.13) for $\widehat{F}_K(s)$. Since $F_K = F_r r_K + F_w w_K$, its transform can be evaluated directly by household shifted solves:

$$\widehat{F}_K(s) = -k_h^\top (sI - \mathcal{L}_{g,0,h})^{-1} \mathbf{B}_{gv,0,h} (sI + \mathcal{L}_{v,h})^{-1} \mathbf{B}_{vp,h} P_K. \quad (6.10)$$

This evaluates the transform of the same fake-news kernel without numerical integration over a truncated two-dimensional time window. The household shifted solves give

$$\begin{aligned} \mathcal{Y}_{v,K}(s) &= -(sI + \mathcal{L}_{v,h})^{-1} \mathbf{B}_{vp,h} P_K, \\ \mathcal{Y}_{g,K}(s) &= (sI - \mathcal{L}_{g,0,h})^{-1} \{ \mathbf{B}_{gp,0,h} P_K + \mathbf{B}_{gv,0,h} \mathcal{Y}_{v,K}(s) \}, \\ m_K(s) &= 1 - k_h^\top \mathcal{Y}_{g,K}(s) \\ &= 1 - k_h^\top (sI - \mathcal{L}_{g,0,h})^{-1} \mathbf{B}_{gp,0,h} P_K - \widehat{F}_K(s). \end{aligned} \quad (6.11)$$

Thus the price-consistency step is a scalar division. Economically, $m_K(s)$ subtracts the capital supplied by households from the capital movement proposed to firms, including both anticipated

policy changes and the direct saving response. It contains the stationary feedback underlying (D.18). The complete half-line SSJ also retains its inherited distribution and initial-boundary correction.

For an input $x = (x_v, x_\xi)$, let (v^0, ξ^0) be the fixed-price resolvent response in (4.15), using $\mathcal{L}_{v,h}, \mathcal{L}_{g,0,h}, \mathbf{B}_{gv,0,h}$. The full feedback correction is

$$\Delta \mathcal{R}_h(s)x = \begin{pmatrix} \mathcal{Y}_{v,K}(s) \\ \mathcal{Y}_{g,K}(s) \end{pmatrix} \frac{k_h^\top \xi^0}{m_K(s)}. \quad (6.12)$$

Construct the baseline projection with $\mathcal{L}_{g,0,h}\Xi_h + \Xi_h\mathcal{L}_{v,h} = -\mathbf{B}_{gv,0,h}$, and integrate this correction as in (4.16). This gives actions of $\mathcal{P}_{s,h}, \mathcal{P}_{u,h}$ without a full eigendecomposition of $\mathbf{A}_{GE,h}$. At N_ω positive-frequency nodes, the real feedback correction has rank at most $2N_\omega$. The household solves, baseline action, and conditioning still determine its actual cost. A zero of $m_K(i\omega)$ prevents this split, while a small value calls for finer accuracy and assessment of proximity to a spectral crossing.

Mapped Gauss–Legendre quadrature with 32, 64, and 128 positive-frequency nodes constructs the projection numerically. At the finest quadrature on the $N_a = 100$ grid, its relative idempotence residual is 1.20×10^{-9} . An SVD of the computed projection supplies an orthonormal basis Q_s for its range. There are 198 retained singular values, all at least approximately one, and the largest discarded singular value is below 9×10^{-11} . The change in the orthogonal range projector between 64 and 128 nodes is 3.94×10^{-8} . These are numerical convergence diagnostics. The present moderate-grid calculation materializes the projection. Larger implementations can use its actions in the projected boundary equations without storing a full stable basis.

With N_ω positive-frequency quadrature nodes, the real approximation to $\mathcal{P}_s - \mathcal{P}_s^0$ has rank at most $2n_p N_\omega$. Its thin factors can be cached or streamed, with storage proportional to $(n_v + n_g)n_p N_\omega$ when cached after spatial realization, where n_v, n_g are the value and distribution dimensions. This bound concerns the quadrature correction. The exact correction need not have finite rank, and the cost of applying Ξ remains.

The $q = n - 2$ distribution coordinates of the calibrated fixed-labor economy are predetermined, while the value response is free to adjust. Use the projection actions from Step 6 to solve directly for s_0 , as in (A.3). Finding a solution establishes compatibility for the specified forcing and inherited distribution. Uniqueness additionally requires $\ker \mathcal{P}_{u,h} \cap \ker \Gamma_h = \{0\}$: no nonzero stable initial adjustment can leave all predetermined coordinates unchanged. Define the restricted initial-data map $\Gamma_{s,h} = \Gamma_h|_{\text{Ran } \mathcal{P}_{s,h}}$, the grid counterpart of the stable-trace map in Theorem 2. Existence for every initial-data mismatch requires this map to be onto $\mathbb{R}^q$.

Table 2 reports the boundary test. The number of stable directions equals the number of predetermined coordinates, and $\Gamma_{s,h}$ has full rank on both grids. Thus both the kernel and cokernel defects vanish numerically. Each admissible initial distribution and forcing has one compatible stable adjustment in this finite-grid model.

Algorithm 1 Applying the equilibrium split with fake-news household factors

Require: Consistent mass-zero household matrices $\mathcal{L}_g, \mathcal{L}_v, \mathbf{B}_{vp}, \mathbf{B}_{gv}, \mathbf{B}_{gp}, P_g$, an action of Ξ , and numerical error tolerances.

- 1: Verify household stability. Assess $M(i\omega)$ on the imaginary axis using interval and high-frequency bounds when certification is required. (See Appendix A.3, the weighted adjoints and norm scaling in (A.6).)
- 2: Choose frequency quadrature nodes and weights. Factor or precondition the shifted household matrices and form $\mathcal{Y}_v, \mathcal{Y}_g, M$ at those nodes.
- 3: **for** each input profile x **do**
- 4: Apply $\mathcal{P}_s^0 x$ using (4.12).
- 5: At each node solve (4.15) and accumulate the weighted real feedback correction in (4.16).
- 6: Return the approximate $\mathcal{P}_s x$ and $\mathcal{P}_u x = x - \mathcal{P}_s x$.
- 7: **end for**
- 8: Implement adjoint actions from the same factors. Refine quadrature and solve tolerances and check projection, invariance, and boundary residuals.

Ensure: Actions of the equilibrium spectral projections to the assessed tolerance, conditional on the stated spectral and numerical bounds.

Table 2: Calibrated Aiyagari determinacy from the fake-news projection

N_a	n	q	n_s	$\text{rank } \Gamma_{s,h}$	$\sigma_{\min}(\Gamma_{s,h})$
50	100	98	98	98	1.69674×10^{-5}
100	200	198	198	198	8.72229×10^{-6}

Notes: N_a counts asset nodes per income state. Both income shares are fixed, giving $q = n - 2$. The stable-space norm is inherited from the Euclidean norm in value and cell-probability coordinates. The reported diagnostics evaluate the restricted map in orthonormal coordinates, as described in Appendix C.10.

6.3 Impulse responses and exact continuation

We now apply the continuation condition from Section 5.2 inside an aggregate SSJ calculation. Theorem 3 establishes equivalence of the continuous-time prefix formulation and the half-line equilibrium. Time discretization makes the implemented SSJ an approximation to that common response. The matching date is a computational interface, distinct from the announcement date and the endpoint displayed in an IRF figure.

For the productivity experiment, set $\delta Z(t) = Z_0 e^{-\nu t}$, with $Z_0 = 0.001$, $\nu = 0.2$, and $\xi_0 = 0$. This is a one-percent initial TFP increase. Return to (6.7). Partition the stable basis by values and distribution coordinates:

$$Q_s = \begin{pmatrix} V_s \\ G_s \end{pmatrix}, \quad \Gamma_h Q_s = G_s. \quad (6.13)$$

If the determinacy test finds $n_s = q$ and nonsingular G_s , where q counts the admissible distribution coordinates, define $Q = V_s G_s^{-1}$. In the fixed-income-share calculation of Section 6.2, $q = n - 2$. With only a total-mass restriction, $q = n - 1$. With no remaining shock, the exact terminal condition is $v_h(T) = Q\xi(T)$. Notice that the very matrix whose invertibility established a unique response to the initial distribution now converts a future distribution into its equilibrium continuation value. The two calculations use the same object.

For the persistent productivity shock, solve on the unstable subspace

$$X_{u,h} = (A_{u,h} + \nu I)^{-1} P_{u,h} b_{Z,h}, \quad r_T = -X_{u,h} \delta Z(T). \quad (6.14)$$

The inverse exists because $A_{u,h}$ has eigenvalues with positive real parts and $\nu > 0$. Partition $X_{u,h} = (X_v^\top, X_\xi^\top)^\top$, and set $q_Z = -X_v + QX_\xi$. The affine terminal relation becomes

$$\boxed{v_h(T) = Q\xi(T) + q_Z \delta Z(T).} \quad (6.15)$$

The additional term matters even when the shock is small at T : it incorporates the households' response to its entire remaining path. Setting it to zero prematurely changes the economic experiment. In the absence of a graph representation, use the projection condition (6.1) directly.

Once the matched prefix has delivered $\xi(T)$, the tail can be propagated without adding any future time grid. Define

$$A_{cl,h} = \mathcal{L}_{g,0,h} + B_{gp,0,h} P_K k_h^\top + B_{gv,0,h} Q, \quad b_{cl,h} = B_{gp,0,h} P_Z + B_{gv,0,h} q_Z. \quad (6.16)$$

The joint distribution and shock state then follows

$$\begin{pmatrix} \xi(T+s) \\ \delta Z(T+s) \end{pmatrix} = \exp \left[s \begin{pmatrix} A_{cl,h} & b_{cl,h} \\ 0 & -\nu \end{pmatrix} \right] \begin{pmatrix} \xi(T) \\ \delta Z(T) \end{pmatrix}, \quad s \geq 0. \quad (6.17)$$

Recover v_h from the same affine graph, $\delta K = k_h^\top \xi$, and $\delta p = P_K \delta K + P_Z \delta Z$. Other aggregates follow from the policy and distribution responses. This formula retains all modes of the fixed spatial realization, including polynomial-exponential behavior if eigenvalues coincide. It is not a fit of a few exponentials to the end of a truncated IRF.

In a large model, explicitly forming Q may be too expensive. The projection actions in (4.16), with (6.12) for Aiyagari, allow direct use of $\mathcal{P}_{u,h}(v_h(T), \xi(T)) = r_T$. A graph action $Q\xi$ can also be obtained by solving this constraint for v with zero future forcing when the graph exists. This requires neither a stored Q nor an explicit stable basis. It still requires controlled household solves, quadrature, and an adequately conditioned boundary solve. Section 5.2 incorporates the projection directly into the prefix calculation. The exact target remains (6.1).

For the Aiyagari graph representation (6.15), the terminal condition is $v_T = Qg(T) + q_Z\delta Z(T)$, written here in the distribution coordinates of (3.8). On a grid, the same calculation uses $\xi, \mathcal{L}_{g,0,h}, \mathbf{B}_{gv,0,h}$ and the corresponding reduced maps. Using $g(T) = g^E(T) + \Xi_T v_T$ from (5.9) gives

$$(I - Q\Xi_T)v_T = Qg^E(T) + q_Z\delta Z(T). \quad (6.18)$$

This equation imposes consistency: terminal values must be those implied by the distribution they themselves help produce. The Aiyagari intercept $q_Z\delta Z(T)$ specializes the future-forcing term $h_v(T) - Qh_g(T)$ in Section 5.2. Under the household stability assumptions defining Ξ in (D.6), the same stationary response operator also gives

$$\Xi_T = \Xi - e^{\mathcal{L}_g T} \Xi e^{\mathcal{L}_v T}.$$

Thus the fake-news boundary correction and the terminal response use the same policy-to-distribution propagation. This identity permits reuse of an available representation of Ξ . Forming a dense matrix is not required. If $I - Q\Xi_T$ is invertible, solve this interface equation for each trial price path. Its matrix or an appropriate solver can be reused across Jacobian actions at the same steady state and matching date. The inverse is an additional condition for this particular elimination. The existence of a tail projection alone does not establish it.

Substituting the resulting v_T gives the aggregate response

$$\delta Y(t) = \delta Y^E(t) + \mathcal{O}_v e^{\mathcal{L}_v(T-t)} v_T + \mathcal{O}_g \int_0^t e^{\mathcal{L}_g(t-r)} \mathbf{B}_{gv} e^{\mathcal{L}_v(T-r)} v_T dr. \quad (6.19)$$

For a Jacobian action, set $g_0 = 0$ and the exogenous shock deviation to zero, apply a trial δp , and perform (D.21), (6.18), and (6.19). For the forcing term, set the trial prices to zero and retain the given initial state and shock inputs, including b_v, b_g in the reference response. Thus both the equilibrium Jacobian and its right side receive the tail correction. Adding the unchanged prefix firm and market-clearing equations produces the matched aggregate operator $\mathcal{S}_T$ in (5.7) whenever that Schur representation is valid. The household calculation is conditional on prefix prices and on equilibrium continuation after T . It need not clear prefix markets until the outer SSJ solve is completed.

The correction factors through the terminal value vector. It is finite rank on a fixed finite spatial grid, but its rank can be as large as that vector's dimension. A low-dimensional aggregate price vector therefore does not by itself make construction of Q or solution of the interface equation cheap. Algorithm 1 instead supplies projection actions through thin frequency-response factors, with a separate baseline household action and assessed quadrature error. This is a constructive alternative to storing the graph, not a bound on the rank or conditioning of the entire terminal correction.

To obtain the plotted aggregate IRFs, recover the cell-mass perturbation $\delta m_h(t) = Q_m \xi(t)$, and use the policy derivatives from Step 2:

$$\begin{aligned}\delta K(t) &= k_h^\top \xi(t), & \delta p(t) &= P_K \delta K(t) + P_Z \delta Z(t), \\ \delta c_h(t) &= B_{cv} v_h(t) + B_{cp} \delta p(t), & \delta C_{\text{agg}}(t) &= \bar{m}_h^\top \delta c_h(t) + \bar{c}_h^\top Q_m \xi(t).\end{aligned}$$

Here B_{cv}, B_{cp} denote the grid policy derivatives, $\bar{m}_h$ is the stationary cell-probability vector, and $\bar{c}_h$ is stationary household consumption. Aggregate consumption changes both because households change their policies and because population mass moves across states with different consumption levels. The first component of δp is the interest-rate response. Figure 3 plots $100\delta K/\bar{K}$, $100\delta C_{\text{agg}}/\bar{C}_{\text{agg}}$, and $10^4\delta r$, where $\bar{C}_{\text{agg}} = \bar{m}_h^\top \bar{c}_h$. Inherited capital cannot jump, while consumption rises at the announcement. Subsequent saving builds capital and eventually lowers its return below steady state.

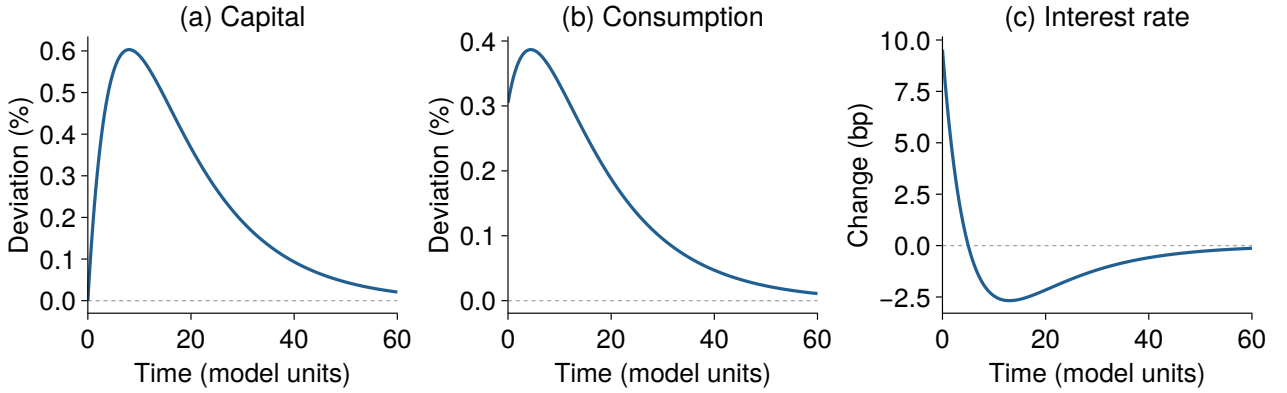


Figure 3: Impulse responses to a one-percent initial productivity increase

Notes: $N_a = 100$, $\delta Z(t) = 0.001e^{-0.2t}$, and $\xi_0 = 0$. Capital and consumption are percentage deviations from their steady-state levels. Interest is a basis-point change. The curves are the continuous-time linear responses in (D.20), sampled every 0.25 model-time units. The displayed endpoint is a plotting window, not a terminal condition. Admissibility is imposed on the infinite horizon.

For the Aiyagari productivity experiment, the same projection and forced trace used to construct Figure 3 give

$$\mathcal{P}_{u,h} \begin{pmatrix} v_h(T) \\ \xi(T) \end{pmatrix} = r_f e^{-\nu T}. \quad (6.20)$$

The scalar exponential scales the forced unstable trace. The stable part of the response is still selected by the prefix equations and inherited distribution. Since the boundary test in Section 6.2 passes, this condition also admits the graph $v_h(T) = Q\xi(T) + q_Z\delta Z(T)$, as derived in (6.15).

The numerical SSJ represents prices as constant on each prefix cell and collocates firm pricing at the cell midpoint. Household integration within each cell is exact, but the representation of the equilibrium price path still introduces time-discretization error. Table 3 compares the computed capital response with the continuous-time projected reference on the same 100-node asset grid.

At $T = 20$, halving the time step roughly quarters the error with the continuation condition. These are numerical refinement results. Convergence to the fixed-grid reference requires a

Table 3: Capital-path accuracy of the aggregate SSJ calculation

Matching date T	Time step Δt	With continuation	With $v_h(T) = 0$
20	1.00	0.006492	0.141352
20	0.50	0.001699	0.146257
20	0.25	0.000435	0.148659
40	0.50	0.001699	0.035810

Notes: $N_a = 100$. Errors are the maximum absolute difference at prefix cell midpoints from the continuous-time projected response, in percentage points of steady-state capital. The two calculations use the same time grid. The continuation condition retains both endogenous future adjustment and the remaining productivity shock.

consistent, stable prefix discretization, with projection and solve errors tending to zero. Convergence to a continuous-state economy additionally requires spatial and differentiated consistency and appropriate spectral stability. By contrast, imposing $v_h(T) = 0$ at a fixed T changes the boundary-value problem: reducing the time step alone does not remove that error.

Figure 4 varies the matching date while holding the time step fixed. The exact continuation retains both the endogenous future adjustment and the remaining productivity shock at every date. Moving the artificial zero terminal value farther away reduces its boundary distortion, but its accuracy depends on the horizon. At $T = 80$ and 100 , the remaining total discrepancies are of the same order as the prefix discretization error. That total need not decrease monotonically with T . In the matched formulation, the continuation itself introduces no such horizon error. Remaining differences reflect the numerical prefix and tail evaluations. Appendix C.10 supplies implementation checks and reproduction commands, and Appendix D.9 isolates the same mechanism in an exactly solvable example.

What is inherited from continuous geometry? The logical order is now complete. The household generator and its adjoint define behavior and distributional propagation. Their policy derivative produces mass-zero fake news. Market clearing closes the household responses into a GE generator or its SSJ Schur complement. The stable space of that closure both tests determinacy and determines the future continuation. DEC preserves the first set of identities exactly on the mesh, allowing the latter two calculations to use that same model. Under the required convergence assumptions, refinement then relates this internally consistent numerical construction to the continuous-state theory.

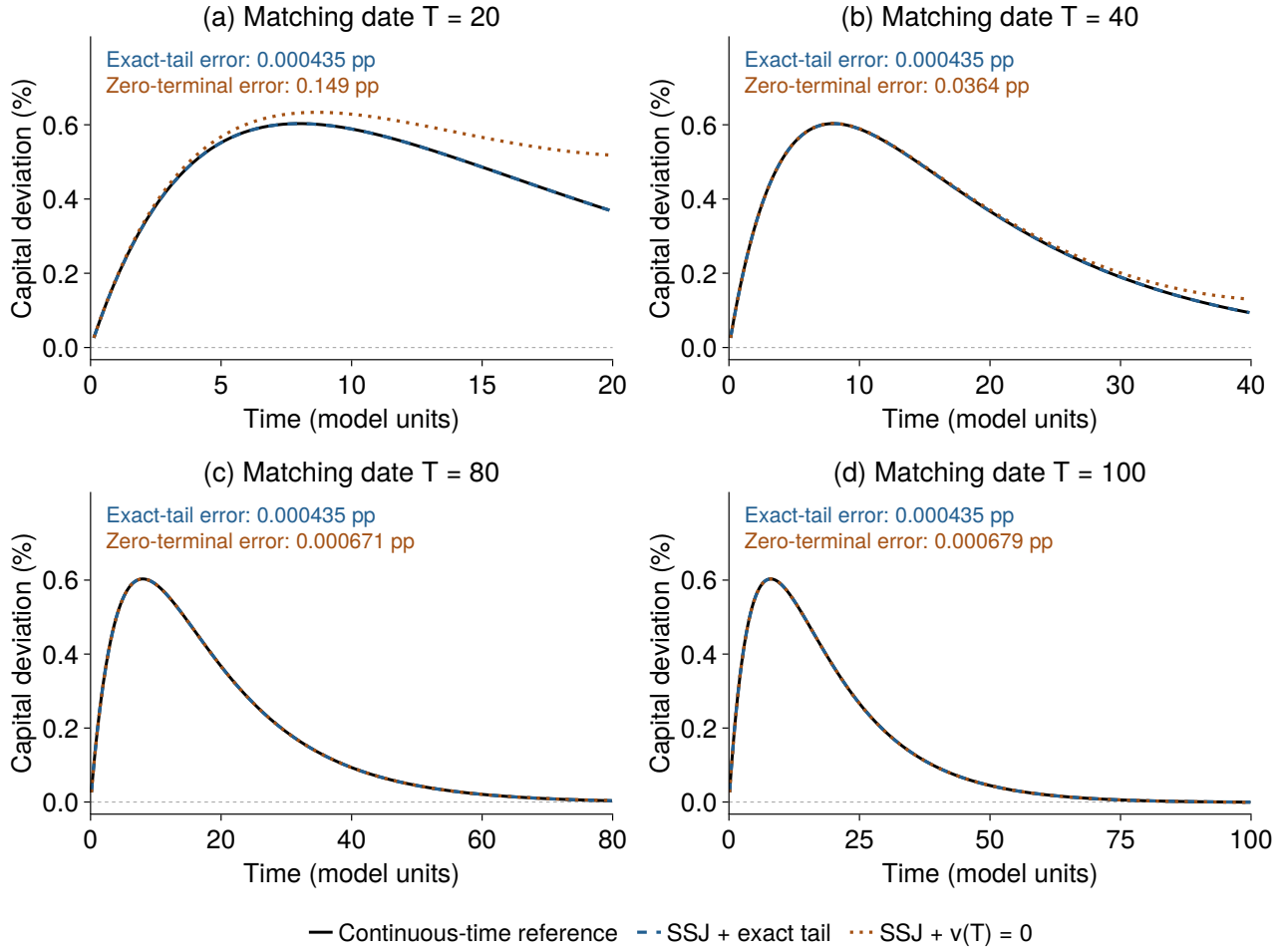


Figure 4: Capital responses across matching dates

Notes: The calibration and productivity shock are those of Figure 3. Each panel compares the continuous-time projected reference with the aggregate SSJ using exact continuation and with the SSJ imposing $v_h(T) = 0$, for $T = 20, 40, 80, 100$. All calculations use $N_a = 100$ and $\Delta t = 0.25$. The reference is sampled at the same cell midpoints. The endpoint of each panel is its matching date. Capital responses are percentage deviations from steady state. Numerical errors remain even where the continuation and reference curves are visually indistinguishable. Panel error labels report the maximum absolute discrepancy from the reference at prefix cell midpoints, in percentage points.

7 Conclusion

The household policy response, the distribution response, and the general-equilibrium price response form one economic feedback system. Its continuous-time sequence-space representation is a boundary-value relation linking optimal decisions, mass-preserving transport, and market clearing. Its geometry separates three requirements: admissible dynamics, compatible shocks, and a unique intersection with predetermined initial data. Under closed-range and dichotomy assumptions, these requirements reduce to kernel and cokernel conditions on a stable-trace map. The same construction produces an exact infinite-tail boundary condition.

This connection provides a precise route to eliminating time-horizon truncation error. A finite prefix with its exact admissible continuation represents the half-line equilibrium, including future forcing. Analytical fake-news household factors also construct the finite-grid spectral split through a frequency-dependent price solve. Their dated responses evaluate the reference household Jacobian. A terminal feedback solve supplies the continuation-value correction within each Jacobian action. The calibrated Aiyagari calculation uses the direct projected IRF as a continuous-time reference on the same spatial grid. Its time-refinement and matching-date comparisons illustrate the distinction between numerical approximation and an artificial terminal restriction. Valid elimination of internal paths retains aggregate sequences as the equilibrium unknowns and preserves the same closure defects. DEC provides a compatible realization of the transport component, while analytic policy responses and matrix-free methods supply possible computational gains.

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